Numerical modeling of elastodynamic radiation and scattering
Milos Savic*, Delft University of Technology, The Netherlands; and Anton M. Ziolkowski, Edinburgh University, U.K.

Abstract

This paper presents a study on two problems: the two-dimensional distributed surface load problem, and the scattering of elastodynamic waves from fractures. The analysis is done with the aid of the finite-difference technique.

If the dimensions of a surface mechanical source (vibrator or piezoelectric transducer) are not small compared to the wavelength, one should not use the point source or plane wave representation when modelling radiation from such sources. Here we demonstrate the solution of the uniformly distributed surface load problem using the finite-difference (FD) technique.

The scattering of transient elasto-dynamic waves from a fracture whose extent is large compared with the wavelength and whose width is small compared with the wavelength is one of the classical problems in seismology and non-destructive testing (NDT). Many researchers have provided analytical solutions based on different approximations for the unknown field (displacement or particle velocity) scattered from an idealized half-plane or the a strip of finite extent. Again, we demonstrate the full wavefield solution using the finite-difference technique.

The technique presented here is aimed for the interpretation of seismic data from hydraulic fracturing experiments.

Introduction

The numerical scheme is based on the finite-difference solution of the first-order coupled system of elastodynamic equations of motion and constitutive laws given as particle velocities and stresses on a staggered-grid. The scheme was developed by Madariaga (1976) to model an expanding circular fault. It was later adopted by Virieux (1984, 1986) for two-dimensional forward modelling of SH- and P-SV waves.

Levander (1988) presented a P-SV isotropic velocity-stress formulation that is fourth-order accurate in space and second-order accurate in time, using the known field (displacement or particle velocity) scattered from an idealized half-plane or the a strip of finite extent. Again, we demonstrate the full wavefield solution using the finite-difference technique.

The technique presented here is aimed for the interpretation of seismic data from hydraulic fracturing experiments.

Basic Equations

A system of coupled, first-order partial differential equations used for the finite-difference discretisation is given by:

\[ \partial_t \tau_{11} = c_{11} \partial_t v_1 + c_{13} \partial_t v_3, \]

\[ \partial_t \tau_{13} = c_{66} (\partial_t v_3 + \partial_t v_1), \]

\[ \partial_t v_1 = \frac{1}{\rho} (\partial_t \tau_{11} + \partial_t \tau_{13}), \]

\[ \partial_t v_3 = \frac{1}{\rho} (\partial_t \tau_{13} - \partial_t \tau_{11}). \]

In which \(\tau\) is the stress, \(v\) is the particle velocity, \(\rho\) is density, \(c\) is the stiffness, and 2-coordinate has been eliminated. The finite-difference solution of the initial value problem necessarily requires that the updating of particle velocities is uncoupled in time from the evaluation of stresses, otherwise the system would be unsolvable. The spatial uncoupling (grid staggering) is not required, but it is desirable since the overall performance is improved due to the halving the spatial interpolation interval.

The leading term of error in the staggered-grid scheme is four times smaller than in case of non-staggered grid schemes. This is because the error is proportional to the square of the sampling interval, and the interval was shortened by one-half.

The discretisation of the system is carried out using explicit staggered-grid finite-difference operators that are second-order accurate in time and fourth-order accurate in space – \(O(\Delta t, h^4)\), where \(\Delta t\) is the time discretisation step, and \(h\) is the space discretization step, which is chosen equal in both \(x_1\) and \(x_3\) directions.

The analytical expression for a source function is taken to be the first derivative of a Gaussian:

\[ f(t) = -2a(t-t_0)e^{-a(t-t_0)^2} H(t-t_0), \]

where \(a\) is a constant which controls the frequency content of the wavelet, and \(H(t-t_0)\) is a Heaviside operator.

Distributed Normal Load Problem

The surface load problem (Lamb’s Problem) is a classical problem in seismology. Many researchers have studied it, but the fundamental analytical solution was provided by Lamb (1904). Elegant transient solution for two-dimensional problem was recently given by Stam (1990) using Cagniard-de Hoop technique.

If dimension of the surface mechanical source (vibrator or piezoelectric transducer) is large compared to the wavelength, one should not use the point source or plane wave representation when modelling radiation from such sources. The nearest approximation for such mechanical radiators in two dimensions resembles strip geometry where the diameter of the radiator is
equal to the width of the strip. The excitation of large mechanical radiators is commonly modelled by uniformly distributed stress strip load acting on the traction free surface of elastic half-space.

In case of the normal stress applied on the surface of elastic half-space the following boundary conditions apply:

$$
\tau_{33}(x,0,t) = \begin{cases} 
  f(t), & 0 \leq |x| \leq a \\
  0, & |x| > a 
\end{cases}
$$

where $f(t)$ is the source intensity and $a$ is the strip half-width.

In the near-field the presence of the body waves, head waves, plane P-wave component directly below the source, and surface waves is expected. Figure 1 shows schematically the theoretical prediction of the radiation from an infinitely long strip (two-dimensional distributed load problem) acting normally on the surface of an ideal elastic half-space (adopted from Stam (1990), Figure 16). The compressional, shear, and head wave components are denoted as P, S, and H, respectively. The waves originating from the left edge of the strip source have superscript (-a), and those originating from the right edge of the strip have superscripts (a). Figure 2 shows a snapshot of the $v_3$ particle velocity component calculated with the finite-difference method assuming the same normal strip model as in the theoretical case. Note the existence of pronounced Rayleigh wave components propagating away from the edges of the strip in both directions along the surface. The Rayleigh wave components are denoted R with corresponding (-at), (-a+), (a-) and (a+) superscripts. Again (-a) and (a) denote the side of the strip the waves originating from and arrows ($\rightarrow$) and (t) indicate the direction of propagation.

Scattering from a half-plane

The problem of the elastodynamic scattering and diffraction from an infinitely rigid or infinitely compliant strip is another classical problem in seismology and non-destructive testing. This problem has been treated by many researchers. The steady state two-dimensional plane wave problem was treated by Maue (1953), de Hoop (1958) gave a solution for the 2D transient plane wave problem, Achenbach et al. (1982) gave a solution for the 3D problem using the Geometric Theory of Diffraction (GTD) at the high frequency limit. Tan (1975) and du Cloux (1986) gave integral equation solutions for time-harmonic and transient point sources for the two-dimensional and three-dimensional problems, respectively, by means of the Cagniard-de Hoop technique.

In analytical solutions a variety of approximations are being applied in order to get a quantitative description of the field scattered from the fracture of the finite extent. The kernel term in the integral representation of the scattered field is often expanded in cosine or sine series (Tan, 1977) or in the Chebyshev polynomials (van der Berg, 1982, van der Hidjen and Neerhoff, 1984). Yet, numerical techniques are still required to come to the final results.

We demonstrate that the problem of scattering and diffraction from any type of inclusion (rigid, compliant, fluid-filled) in any type of surrounding medium (isotropic, anisotropic, homogeneous, heterogeneous) can be handled with ease using finite-difference technique (see also Langenberg et al. 1993).

Figure 3 shows the scheme of theoretically predicted wave fronts for diffraction of a plane P-wave by a perfectly rigid half-plane (from de Hoop, 1958). In this configuration no surface wave propagates over the strip. Figure 4 shows a snapshot of the $v_3$ component of the particle velocity obtained with a finite-difference numerical solution of the two-dimensional (P-SV) elastic wave equation. The set-up is the same as in the top display. Figure 5 shows the scheme of theoretically predicted wave fronts for diffraction of a plane P-wave by a perfectly compliant half-plane (from de Hoop, 1958). In this configuration the surface wave exists. Figure 6 shows a snapshot of the $v_3$ component of the particle velocity obtained with a finite-difference numerical solution of the two-dimensional (P-SV) elastic wave equation. The set-up is the same as in the top display except that the fracture filled with fluid was modelled. The boundary conditions then require the shear stress to vanish and normal stress to be continuous. All the features present in the top display are also present in the bottom display, except that there is no shadow zone, since the P-wave has been partly transmitted through the slip. Part of the transmitted compressional energy is mode-converted to transmitted S-wave.

Application

We have shown that the finite-difference technique can be applied to the problems of distributed normal stress load and the scattering of elastodynamic waves from a fracture. We will demonstrate the application of the technique to the interpretation of seismic data from hydraulic fracturing experiments in the field (Wills et al., 1992; Meadows and Wintherstein, 1994) and in the laboratory (Savic et al, 1993).

Acknowledgement

This research has been partly sponsored by the European Union under project number TH/03.31I/89-NL.

References


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**Figure 1** The theoretical prediction of the wave fronts in the case of a normal uniform strip load (adopted from Stam, 1990, Figure 16).

**Figure 2** A snap-shot of the $p_3$ component of the particle velocity for the case of a normal uniform strip load acting on the surface of an elastic half-space, calculated with the finite-difference technique.
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Figure 3 Wave fronts for diffraction of a plane P-wave by a perfectly rigid half-plane (adopted from de Hoop 1958, Fig. 7).

Figure 4 A snap-shot of the $v_3$ component of the particle velocity for a case of the plane P-wave diffracted from the perfectly rigid half-plane. Compare to the analytical prediction shown in Figure 3.

Figure 5 Wave fronts for diffraction of a plane P-wave by a perfectly compliant (stress-free) half-plane (adopted from de Hoop 1958, Fig. 10).

Figure 6 A snap-shot of the $v_3$ component of the particle velocity for a case of the plane P-wave diffracted from the fluid filled fracture. The slip boundary conditions across the strip have been used. Note that there is no shadow zone since the P-wave is transmitted through slip (event 7). Part of the transmitted compressional energy is mode converted to a transmitted S-wave (event, 8). Also note a pronounced interface wave.