Interferometric modelling of wave propagation in inhomogeneous elastic media using time-reversal and reciprocity

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ABSTRACT

Many applications involving (acoustic, electromagnetic or elastic) wave-propagation, such as waveform inversion, imaging and survey evaluation and design require a large number of modelled solutions to the wave equation. The most complete methods of solution, which accurately model high-order interactions between scatterers in a medium, typically become prohibitively expensive for realistically complete descriptions of the medium, source and receiver geometries, and hence for solving realistic problems. We present a methodology providing a new perspective on modeling and inversion of wave propagation in generally inhomogeneous media based on time-reversal invariance and reciprocity. The approach relies on a representation theorem of the elastodynamic wave equation to express the Green function between points in the interior of a medium as an integral over the response in those points due to sources on a surface surrounding the medium. Following a predictable initial computational effort, Green’s functions between arbitrary points in the medium can be computed as needed using a simple cross-correlation algorithm. The approach is first illustrated on an acoustic model consisting of isotropic points scatterers embedded in a homogeneous background medium. This is followed with an elastic example for a region of the Pluto model. We end with a discussion of the computational aspects of the new method and the implications for modelling and inversion.

INTRODUCTION

Many applications in diverse fields such as communications analysis, waveform inversion, imaging, survey and experimental design, and industrial design, require a large number of modelled solutions of the wave equation in different media. The most complete methods of solution, such as finite differences (FD), which accurately model all high-order interactions between scatterers in a medium, typically become prohibitively expensive for realistically complete descriptions of the medium and geometries of sources and receivers, and hence for solving realistic problems based on the wave equation. van Manen et al. (2005) showed that the key to breaking this apparent paradigm for acoustic media lies in a basic reciprocity argument in combination with recent theoretical advances in the fields of time-reversed acoustics (Derode et
al., 2003) and seismic interferometry (Schuster, 2001; Wapenaar et al., 2004; Wapenaar, 2004; Weaver and Lobkis, 2001). Here, we extend that work to elastic media and fully explore the theoretical and computational aspects of the method, and the implications this work may have on modelling and inversion of the wave equation in the future.

In time-reversed acoustics, invariance of the wave equation for time-reversal can be exploited to focus a wave field through a highly scattering medium on an original source point (Derode et al., 1995). Cassereau and Fink (1992, 1993) realized that the acoustic representation theorem (Wapenaar and Fokkema, 2004) can be used to time-reverse a wave field in a volume by creating secondary sources (monopole and dipole) on a surface surrounding the medium such that the boundary conditions correspond to the time-reversed components of a wave field measured there. These secondary sources give rise to the back-propagating, time-reversed wave field inside the medium that collapses onto itself at the original source location. Note that since there is no source term absorbing the converging wave field, the size of the focal spot is limited to half a (dominant) wavelength in accordance with diffraction theory (Cassereau and Fink, 1992). The diffraction limit was overcome experimentally by de Rosny and Fink (2002) by introducing the concept of an “acoustic sink”.

In interferometry, waves recorded at two receiver locations are correlated to find the Green function between the locations. Interferometry has been successfully applied to helioseismology (Rickett and Claerbout, 2000), ultrasonics (Weaver and Lobkis, 2001) and exploration seismics (Wapenaar et al., 2004). Recently it was shown that there exists a close link between the time-reversed acoustics and interferometry disciplines when Derode et al. (2003) analyzed the emergence of the Green function from field-field correlations in an open scattering medium in terms of time-reversal symmetry. The Green function can be recovered as long as the sources in the medium are distributed forming a perfect time-reversal device. A rigorous proof for the general case of an arbitrary inhomogeneous elastic medium with observation points at the free surface was presented by Wapenaar (2004).

Central to the new modelling method is the Kirchhoff-Helmholtz integral. The Kirchhoff-Helmholtz integral is also the basis for many seismic processing algorithms. Hilterman (1970, 1975), Trorey (1970) and Berryhill (1977) used the Kirchhoff-Helmholtz integral to model the reflection and diffraction response of complicated geological structures embedded in homogeneous acoustic background models. This work was generalized to laterally inhomogeneous layered elastic media by Frazer and Sen (1985), who also discussed various asymptotic ap-
proximations of the Kirchhoff-Helmholtz integral to allow fast numerical evaluation. Schneider (1978) showed how the Kirchhoff-Helmholtz integral naturally leads to an integral formulation for migration of CDP stacked data in two- and three-dimensions under the “exploding reflector” model (Loewenthal et al., 1976). The elastodynamic version of the Kirchhoff integral has also been used as a boundary condition in reverse-time (finite-difference) migration (Hokstad et al., 1998; Mittet, 1994) and in the finite-difference injection method proposed by Robertsson and Chapman (2000) to efficiently compute FD seismograms after model alterations. Wapenaar and Haimé (1990) derive a modified elastic Kirchhoff-Helmholtz integral for (inverse) extrapolation of down- and upgoing P- and S-waves.

In migration and forward modelling using the Kirchhoff-Helmholtz integral, often the Green functions used are one-way propagators or asymptotic approximations to limit the computational cost. This limits its use in modelling and migration to a finite number of discrete events, primaries reflected or multiply scattered from a few particular discontinuities. In addition, the boundary conditions are often chosen such that either the traction or the wave field itself disappears on the boundary surrounding the medium (i.e., the surrounding medium is perfectly soft or perfectly rigid, respectively) and contributions over the distant hemisphere usually neglected, in the limit of infinite radius with a reference to Sommerfeld’s radiation condition [e.g., Pao and Varatharajulu (1976)].

Here, we focus on an application of the elastic Kirchhoff-Helmholtz integral to modelling based on time-reversal that accurately models all high-order multiple scattering (including free-surface related multiples). In such a case, the boundary conditions on the surrounding surface cannot simply be chosen to the modelers advantage. Moreover, because back-propagating Green functions are used in the elastic Kirchhoff-Helmholtz integral, the contributions from the (even distant) hemisphere cannot be neglected since they do not satisfy Sommerfeld’s radiation condition. In fact, it will be shown that the usual surface contribution of the Kirchhoff-Helmholtz integral vanishes in the case of a surface with homogeneous boundary conditions (e.g., the free surface) and that the main contribution is from the remainder of the surrounding surface.

The paper is organized as follows. First, we show how the elastodynamic representation theorem can be used to time-reverse a wavefield in a volume and explain how this relation can be turned into an efficient and flexible forward modelling algorithm using a simple reciprocity argument. This extends the work by van Manen et al. (2005) from acoustic to elastodynamic
wave propagation. Next, we illustrate the method for a simple 2D model of three isotropic point
scatterers embedded in a homogeneous acoustic background medium, followed by an example
for a more complicated inhomogeneous elastic medium. Finally, we discuss some important
computational aspects of the new modelling algorithm and possible synergies with methods of
inversion for medium properties.

THE ELASTODYNAMIC REPRESENTATION THEOREM

The basis of our interferometric modelling method is the elastodynamic representation theo-
rem. Representation theorems are usually derived from more general reciprocity theorems and
are closely related. A reciprocity theorem relates two independent acoustic, electromagnetic or
elastodynamic states that can occur in the same spatio-temporal domain, where a state simply
means a combination of material parameters, field quantities, source distributions, boundary
conditions and initial conditions that satisfy the relevant wave equation. In its most general
form it relates a specific combination of field quantities from both states (e.g., pressure and pres-
sure gradient for the acoustic case or particle displacement and traction in the elastodynamic
case) on a surface surrounding a volume, to differences in source distributions, medium param-
eters, boundary conditions or even flow velocities (in case the material is moving) throughout
the volume (Holvik and Amundsen, 2005; Wapenaar and Fokkema, 2004).

Here we consider a special case of the Betti-Rayleigh elastic reciprocity theorem when the
medium in both states is identical and non-flowing. In that case, states (A) and (B) are simply
characterized by the following wave equations (in the space-frequency domain):

\[ \rho \omega^2 u_i^{(A)} + \partial_j \left( c_{ijkl} \partial_k u_i^{(A)} \right) = -f_i^{(A)}, \]
\[ \rho \omega^2 u_i^{(B)} + \partial_j \left( c_{ijkl} \partial_k u_i^{(B)} \right) = -f_i^{(B)}, \]  

(1)

where \( u_i^{(A)} \) and \( u_i^{(B)} \) denote the components of particle displacement for state (A) and (B),
respectively, generated by the components of body force density \( f_i^{(A)} \) and \( f_i^{(B)} \) and where
\( c_{ijkl}(x) \) and \( \rho(x) \) are the stiffness tensor and mass density at location \( x \) in the medium. Note
that Einstein’s summation convention for repeated indices is used. In Appendix A, it is shown
that in this case the Betti-Rayleigh elastic reciprocity theorem becomes:

\[ \int_S \left\{ u_i^{(B)} n_j c_{ijkl} \partial_k u_i^{(A)} - u_i^{(A)} n_j c_{ijkl} \partial_k u_i^{(B)} \right\} \, dS = - \int_V \left\{ f_i^{(A)} u_i^{(B)} - f_i^{(B)} u_i^{(A)} \right\} \, dV. \]  

(2)
A representation integral can be derived from equation (2) by identifying one state with a mathematical or Green state (i.e., a state where the source is a unidirectional point force and the resulting particle displacement is called the elastodynamic Green function (Snieder, 2002; Wapenaar and Fokkema, 2004) and the other with a physical state that can be any wave field set-up by an arbitrary source distribution. Thus we choose state (B) to be the Green state and take $f^{(B)}(x)$ a unit point force at location $x'$ in the $n$ direction: $f^{(B)}(x) = \delta_{in}\delta(x - x')$ and the resulting response the Green tensor: $u^{(B)}_i(x) = G_{in}(x, x')$. We leave state (A) unspecified.

Inserting these expressions, carrying out the volume integral, dropping the superscripts for state (A) and making no assumptions about the the boundary conditions we arrive at:

$$u_n(x') = \int_V G_{in}(x, x')f_i(x) \, dV + \int_S \{G_{in}(x, x')n_jc_{ijkl}\partial_ku_l(x) - u_i(x)n_jc_{ijkl}\partial_kG_{in}(x, x')\} \, dS$$

Finally, applying reciprocity to Green’s tensor and exchanging the coordinates $x \leftrightarrow x'$ and indices $i \leftrightarrow n$ we arrive at the elastodynamic representation theorem:

$$u_i(x) = \int_V G_{in}(x, x')f_n(x') \, dV' + \int_S \{G_{in}(x, x')n_jc_{ijkl}\partial_ku_l(x') - u_n(x')n_jc_{ijkl}\partial_kG_{in}(x, x')\} \, dS'$$.

where $\partial'_kG_{il}(x, x')$ denotes partial derivative of Green’s tensor in the $k$ direction with respect to primed coordinates and $\mathbf{n}$ denotes the normal to the boundary. Thus, the wave field $u_i(x)$ can be computed everywhere inside the volume $V$ once the exciting force $f_n(x')$ inside the volume and the wave field $u_n(x')$ and associated traction $n_jc_{ijkl}\partial_ku_l(x')$ on the surrounding surface $S$ are known.

**TIME-REVERSAL USING THE ELASTODYNAMIC REPRESENTATION THEOREM**

To time-reverse a wave field in a volume $V$, one possibility would be to reverse the particle velocity at every point inside the volume simultaneously. However, Cassereau and Fink (1992) noted that for (partially) open systems (i.e., with outgoing boundary conditions on at least part of the surrounding surface $S$) time-reversal can also be achieved by measuring the wave field and its gradient on the enclosing surface, time-reversing those measurements and letting them act as a (time-varying) boundary condition on the surface $S$. Their approach directly follows from an application of the acoustic representation theorem (i.e., Green’s Theorem or the
Kirchhoff-Helmholtz integral) and is easily extended to elastodynamic wave propagation using equation (4) derived above. Thus, to time-reverse any wavefield $u_i(x)$, due to an arbitrary (unknown) source distribution $f_n(x)$, we simply substitute the complex conjugate of the wave field (phase-conjugation being equivalent to time-reversal), its gradient and its sources into the elastodynamic representation theorem [equation (4)]. This gives:

$$u_i^*(x) = \int_V G_{in}(x,x') f_n^*(x') \, dV' + \int_S \{ G_{in}(x,x') n_j c_{njkl} \partial_k u_n^*(x') - u_n^*(x') n_j c_{njkl} \partial_k G_{il}(x,x') \} \, dS',$$

where a star * denotes complex conjugation. Note that equation (5) can be used to compute the time-reversed wave field (including all high-order interactions) at any location, not just at an original source location. In order for the time-reversal to be complete, the energy converging at the original source locations should be absorbed. Thus, the volume integral on the right hand side of equation (5) corresponds to the wavefield generated by a distribution of “elastic sinks” (de Rosny and Fink, 2002) which destructively interferes with the time-reversed wavefield that propagates through the focii.

Now, say that the wavefield $u_i(x)$ was also originally set-up by a point force source excitation, but at location $x''$ and in the $m$-direction [i.e., $f_i(x) = \delta_{im} \delta(x - x'')$ and $u_i(x)$ is Green’s tensor: $u_i(x) = G_{im}(x, x'')$]. Thus, if we compare equations (4) and (5), it is clear that effectively we are taking the unspecified state to be a time-reversed Green state. Inserting these expressions in equation (5) and carrying out the volume integration gives:

$$G_{im}^*(x, x'') = G_{im}(x, x'')$$

$$+ \int_S \{ G_{in}(x,x') n_j c_{njkl} \partial_k G_{im}^*(x', x'') - G_{nm}^*(x', x'') n_j c_{njkl} \partial_k G_{il}(x,x') \} \, dS'. $$

Equation (6) relates the time-advanced and time-retarded elastodynamic Green functions and was previously derived for the scalar inhomogeneous Helmholtz wave equation and electric and magnetic vector wave fields by Bojarski (1983). Note that the time-retarded (causal) Green function in the r.h.s corresponds to the wavefield generated by the point force elastic sink. In the following the elastic sink will not be modelled – only the integral term in equation (6) will be calculated. Hence, $G_{im}^*(x, x) - G_{im}(x, x)$ will be obtained.

Physically, this means that the converging wavefield will immediately start diverging again after focusing and that the size of the focal spot is limited to half the minimum wavelength
in accordance with diffraction theory (de Rosny and Fink, 2002). Mathematically, the time-retarded Green function has to be subtracted from both sides of equation (6) and the time-reversed wavefield is a solution to the homogeneous wave equation (i.e., without a source term) (Cassereau and Fink, 1992; Oristaglio, 1989).

The latter follows immediately when subtracting the wave equations for the forward and reversed states: the time-advanced and time-retarded Green function satisfy, respectively,

\[ \rho \omega^2 G^*_{im} + \partial_j (c_{ijkl} \partial_k G^*_{im}) = -\delta_{im} \delta(x - x''), \]  

(7)

\[ \rho \omega^2 G_{im} + \partial_j (c_{ijkl} \partial_k G_{im}) = -\delta_{im} \delta(x - x''). \]  

(8)

Subtracting equation (8) from equation (7), we arrive at:

\[ \rho \omega^2 (G^*_{im} - G_{im}) + \partial_j (c_{ijkl} \partial_k (G^*_{im} - G_{im})) = 0. \]  

(9)

Thus, in the absence of an elastic sink, equation (6) becomes:

\[ G^*_{im}(x, x'') - G_{im}(x, x'') = \int_S \{ G_{in}(x, x') n_j c_{njkl} \partial_k G^*_{lm}(x', x'') - G^*_{mn}(x', x'') n_j c_{njkl} \partial_k G_{il}(x, x') \} \, dS'. \]  

(9)

Equation (9) states that by measuring or computing the time-reversed wave field in a second location, \( x \), the Green function and its time-reverse between the source point \( x'' \) and the second point \( x \) are observed. This agrees with other recent experimental and theoretical observations (Derode et al., 2003; Wapenaar, 2004). Using reciprocity: \( G_{ij}(x', x) = G_{ji}(x, x') \), we can rewrite equation (9) so that it involves only sources on the boundary enclosing the medium:

\[ G^*_{im}(x, x'') - G_{im}(x, x'') = \int_S \{ G_{in}(x, x') n_j c_{njkl} \partial_k G^*_{ml}(x'', x') - G^*_{mn}(x'', x') n_j c_{njkl} \partial_k G_{il}(x, x') \} \, dS'. \]  

(10)

Hence, Green’s function between two points \( x \) and \( x'' \) can be calculated once the Green functions between the enclosing boundary and each of these points are known.

A highly efficient two-stage modelling strategy follows from equation (10): first, the Green function terms \( G \) and \( \nabla' G \) are calculated from boundary locations to internal points in a conventional forward modelling phase; in a second inter-correlation phase, the integral is calculated requiring only cross-correlations and numerical integration. Since the computational cost of typical forward modelling algorithms (e.g., FD) does not significantly depend on the number of receiver locations but mainly on the number of source locations, efficiency and flexibility are
achieved because sources need only be placed around the bounding surface, not throughout the volume. The modelled wave field should be stored for each of the boundary sources in as many points as possible throughout the medium. To calculate Green’s function between two points the recordings in the first point due to the dipole sources on the boundary are cross-correlated with the recordings in the second point due to the monopole sources, and vice-versa. The resulting cross-correlations are subtracted and numerically integrated over the boundary of source locations. Unprecedented flexibility follows from the fact that Green’s functions can be calculated between all pairs of points that were defined up front and stored in the initial boundary source modelling phase. Thus, we calculate a partial modelling solution that is common to all Green’s functions, then a bespoke component for each Green function. A flowchart of the new modelling method is given in Figure 1 and is discussed in detail below, in connection with an acoustic isotropic point scattering example.

Special case: time-reversal and modelling of acoustic waves

The time-reversal modelling formula for acoustic waves can be derived similarly, as discussed in detail by van Manen et al. (2005). Here, we simply state their result, valid for partially open acoustic media (i.e., with outgoing, radiation or absorbing boundary conditions on at least part of the surrounding surface):

\[
G_{pr}^*(x, x'') - G_{pr}(x, x'') = 
\int_S \frac{1}{\rho(x')} \left[ G_{pr}(x, x') \nabla' G_{pr}^*(x'', x') - \nabla' G_{pr}(x, x') G_{pr}^*(x'', x') \right] \cdot n \, dS',
\]

where \(G_{pr}(x, x'')\) denotes the Green function for the pressure at location \(x\) due to a point source of volume injection at location \(x''\) and \(\nabla' G_{pr}(x, x'') \cdot n\) denotes the normal derivative of Green’s function with respect to primed coordinates. Thus, the pressure Green function \(G_{pr}(x, x'')\) between two points \(x\) and \(x''\) can be calculated once the Green functions between the enclosing boundary and these points are known. Note that the terms \(G_{pr}(x, x'')\) correspond to simple monopole sources on the surrounding surface whereas the terms \(\nabla' G_{pr}(x, x'') \cdot n\) correspond to dipole sources. This formula will be used in the next section to compute the Green function between points in a 2D acoustic model with three isotropic point scatterers embedded in a homogeneous background medium.
EXAMPLE I: 2D ACOUSTIC ISOTROPIC POINT SCATTERING

The methodology described above will now be explained in more detail using a simple 2D acoustic example. A more realistic elastic model, including strong heterogeneity and interfaces with homogeneous boundary conditions, is discussed in a later section. In Figure 2, three isotropic point-scatterers are shown, embedded in a homogeneous background medium of infinite extent (background velocity $c_0 = 750$ m/s). The point-scatterers are indicated by large black dots. The new method is used to model Green’s functions between arbitrary source and receiver locations in the medium which include all high-order interactions between the isotropic point scatterers.

As indicated in the flowchart in Figure 1, in the first step a boundary enclosing the medium is defined and spanned by source locations. A large number of so-called “points of interest” are also specified. In Figure 2, every second boundary source location is marked with a star. The boundary sources should be spaced according to local Nyquist criteria. The grid of small points are the points where we may be interested in placing a source or receiver later. The number of points of interest should be chosen as large as possible, the only limitation being the waveform data storage capacity. If there are no significant storage constraints and FD is used for simulating Green’s functions, then a reasonable choice would be every fifth FD grid point since for most applications the oversampling of the wavefield by a factor of 5-8 in the FD computations (required for accuracy) is not required – an exception to this could be the case where one is interested in calculating spatial derivatives of Green’s functions computed with the new method. In such a case, the waveforms could be stored (locally) for each FD grid point. In Figure 2, the triangles denote some particular points of interest that we will be looking at later.

In the second step of the initial phase, separate conventional forward modelling runs are carried out for each source on the boundary and the wavefield is stored at all points of interest. In this example, we have used a deterministic variant of Foldy’s method (Foldy, 1945; Groenenboom and Snieder, 1995; Snieder and Scales, 1998) to compute the multiply scattered wavefield for each boundary source. Note that we could have used any method that accurately models multiple scattering (e.g., FD). Our methodology is not restricted to any particular forward modelling method or code. In Figure 2, a snapshot of the early stages of the wavefield is shown for a particular source on the enclosing surface. Thus, in the second step, the interior of the
model is systematically illuminated from the surrounding surface. During or after the simulations for all boundary sources, it is convenient to sort the data into so-called point-of-interest gathers comprising data from all boundary sources recorded at each point of interest. These constitute a common component of all Green’s functions involving that point-of-interest.

In the second, intercorrelation phase, we may now calculate the Green function between any pair of points that were defined beforehand by cross correlation and summation of boundary source recordings. In Figure 2, the triangles denote a subset of points that we could be interested in as part of, e.g., a cross-well survey design experiment.

In Figure 3, panels (a) and (d), the modelled wavefield due to each monopole source on the boundary is shown for two of the points of interest (with coordinates [-50,0] and [50,-50] respectively). Note that, even though there are only three isotropic point scatterers, several multiply scattered waves can easily be identified. Also note the flat event at approximately $t = 0.2$ s. This is the incident wave from each boundary source, scattered isotropically in the direction of the two points of interest by the central scatterer (which is equidistant from each boundary source). In panels (b) and (c), the normal derivative with respect to the boundary has been computed by spatial filtering of the point of interest gathers (a) and (d), respectively, to simulate the response due to dipole sources on the boundary. This is possible since we have outgoing (i.e., absorbing or radiation) boundary conditions on the surrounding surface and hence the pressure and its gradient are directly related (see Appendix B for details).

Calculation of the normal derivative with respect to the boundary source location is completely equivalent to measuring the response due to a dipole source so, alternatively, we could have modelled the required gradient using a second dipole source type. Typically, however, direct modelling would be computationally much more expensive.

Panels (e) and (f) show the trace-by-trace cross correlation of panels (a) and (c) and (b) and (d) respectively. Thus, they form the two terms in the integrand of equation (11). Note that it is difficult to make a straightforward interpretation of the crosscorrelation panels: although equation (11) predicts that the waveform resulting from summation of these crosscorrelations for all boundary sources will be anti-symmetric in time, panels (e) and (f) clearly are not. This is because, at this stage, we still have not carried through the delicate (but stable!) constructive and destructive interference of the back-propagating wavefield. A more thorough analysis of the features of such cross correlation gathers is presented for the second example. In the final step, crosscorrelation gathers (e) and (f) are weighted by $\rho^{-1}$, subtracted and numerically integrated.
(summed) over all source locations. The resulting intercorrelation Green function and a directly computed reference solution are shown in Figure 3(g). The insets show particular events in the waveform in detail. Note that the reference solution has been slightly shifted vertically to show the match between the overlapping waveforms.

To further illustrate the new modelling method, the intercorrelation phase is now applied repeatedly to calculate Green’s functions for the simple cross-well transmission and reflection seismic experiment shown in Figure 2 (source and receiver locations are indicated by triangles). Note that this does not require any additional conventional forward modelling but instead uses the same data modelled in the initial phase. Also note that we could consider a completely different well location, for any combination of point-of-interests (indicated by small dots in Figure 2) as long as they were defined beforehand and the wavefield was stored in those points during the initial modelling phase.

In Figure 4, panels (a) and (b), Green’s functions computed using a conventional forward modeling method and the new method are shown, respectively. The corresponding source-receiver (transmission) geometry is shown in the inset. Note that the amplitudes have been scaled up by a factor 5 to show the weak, multiply scattered events. In panel (c), the difference between the Green functions computed with the two methods is shown and the amplitude differences have been scaled up by an additional factor 10 to emphasize the excellent match. Similarly, in panels (d),(e) and (f), Green’s functions computed using the new method are compared to a reference solution for the reflection setting shown in the inset. Again, amplitude (differences) have been scaled up. In this case by a factor 25 and an additional factor 10, respectively. Note the mismatch in the Green function for the direct wave close to the original source location. This error results from the missing acoustic sink and the bandlimited nature of the synthetics, and agrees with the theory presented in section .

**EXAMPLE II: 2D ELASTIC PLUTO MODEL**

The new methodology is not limited to the particular type of (idealized) multiple scattering experiment considered above (i.e., the case of isotropic point scatterers embedded in a homogeneous background model). As long as the modelling honors the properties of time-reversal invariance and reciprocity, the new methodology can be applied. In this section we give a second example of our method for a model that is more relevant to the exploration seismic
setting. In Figure 5, the compressional wave velocity in a 4.6 x 4.6 km region of the elastic Pluto model (Stoughton et al., 2001) is shown. This model is often used to benchmark marine seismic imaging algorithms. A high velocity (4500 m/s) salt body on the right represents a common imaging challenge. In black, two points of interest (offset 1 km) are shown. The dotted line denotes the boundary with $N_S = 1200$ source locations distributed with a density consistent with the local spatial Nyquist frequency, which means that the wavefield should not be aliased in the point of interest gathers. Outgoing (i.e., radiation or absorbing) boundary conditions (Clayton and Engquist, 1977) are applied right outside the surface enclosing the points of interest to truncate the computational domain.

Forward simulations were carried out for each of the 1200 source locations on the boundary using an elastic FD code (Robertsson et al., 1994) and the waveforms (components of particle displacement) stored at 90,000 points distributed regularly throughout the model. Since we are dealing with the 2D elastodynamic wave equation, at least two forward simulations have to be carried out for each source location: one for each unidirectional point force in mutually orthogonal directions. Figure 6 shows the first 4 seconds of the $G_{11}(x_1, x'_1)$ component of Green’s tensor modelled for the left point of interest (i.e., the horizontal component of particle displacement in the point of interest $x_1$ due to all point-force sources in the horizontal direction on the boundary at locations $x'_1$). For reference, the first source on the boundary is located just below the free surface on the right in Figure 5, at 2.56 m depth in the water layer. The last source on the boundary is located just below the free surface on the left. Note that because of the cross-symmetry of the terms in the integrand in equation (10), no sources are required along the Earth’s free surface, or any interface with homogeneous boundary conditions (e.g., with either vanishing traction [Neumann] or vanishing particle displacement [Dirichlet]). Intuitively, this can be understood from a method of imaging argument: since such interfaces act as perfect mirrors, reflecting all energy back into the volume, an equivalent medium can be constructed which consists of the original medium combined with its mirror in the homogeneous boundary and the homogeneous boundary absent. Since the original boundary with source locations is mirrored as well, the new boundary does completely surround this hypothetical medium and therefore, the sources constitute a perfect time-reversal mirror in the sense described by Derode et al. (2003). Note that when the free surface has topography, although the method of imaging argument breaks down, this property still holds.

An interesting feature of the data, to which we will return later, occurs approximately between sources 200-475, and between sources 1800-2200. These sources are located in the
near-surface of the sedimentary column, just beneath the water layer. The Pluto model in-cludes many randomly positioned, near-surface scatterers, representing complex near-surface heterogeneity that is often observed in nature. Within these two source ranges it is clear that all coherent arrivals are followed by complicated codas that are superposed, resulting in a multiply-scattered signal that builds with time.

Note that according to equation (10) derivatives of the Green function with respect to the source location on the boundary also have to be computed. These terms correspond to the response due to special (deformation rate tensor) sources on the boundary and seem to require additional modelling with such special sources before Green’s functions can be computed using the new method. However, using reciprocity, these terms can also be interpreted as the traction measured on the enclosing boundary due to unidirectional point forces at a particular point of interest. When the surface surrounding the medium has outgoing boundary conditions, the wave field and the corresponding traction are directly related (Holvik and Amundsen, 2005). In Appendix B, it is explained in detail how these properties can be exploited to avoid the need for additional direct modelling. Briefly, when the wavefield is excited on a straight boundary, embedded in a laterally homogeneous medium (where lateral means along the array), the reciprocal components of particle displacement in the point of interest gather are Fourier transformed into the frequency-wavenumber domain and matrix multiplication with an analytical expression gives the corresponding components of traction before transforming back to the space-time domain. When the boundary is curved and/or the medium is not homogeneous along the source array, spatially compact filter approximations can be designed to filter the data in the space-frequency domain using space-variant convolution. This approach is akin to, e.g., decomposition of multi-component seismic data in the common shot domain using spatially compact filters, where it gives promising results (Amundsen et al., 2005; Robertsson and Curtis, 2002; Robertsson and Kragh, 2002; van Manen et al., 2004).

When all stored components of the Green tensor have been retrieved for two points of interest and the corresponding equivalent traction data are computed through spatial filtering, the point of interest gathers are cross-correlated and summed according to equation (10). Note that even before integration, equation (10) requires summation of cross correlation gathers since Einstein’s summation convention for repeated indices is used. This one-to-many, many-to-one relationship could be expected since to time-reverse a wavefield due to, e.g., a point-force in the horizontal direction at a first point and compute the displacement in the vertical direction at a second point, all components of displacement and traction across the boundary are required
due to the source in the first point, as well as the vertical displacement Green functions due to point-forces and deformation rate tensor sources in all orthogonal directions [see equation (9)].

Note that components of traction are cross correlated with components of particle displacement and vice-versa. This has two effects. First, it undoes the temporal integration that is implicit in the cross-correlation and summation process so that result is again a particle displacement Green function [see Snieder et al. (2006) and Snieder (2004) for an explanation in terms of the stationary phase approximation]. Second, it ensures that waves that are incoming and outgoing at the surrounding boundary are correctly separated in the correlation process. For example, Mittet (1994) shows, using the Kirchhoff integral for elastic waves as a boundary condition in reverse-time finite-difference migration, that only when both particle displacement and traction are measured, the time-reversed wavefield is accurately reconstructed on one side of the boundary (inside the volume). Wapenaar and Haimé (1990), using one-way wave equations, derive explicitly which up- and down-going Green quantities in the forward state interact with the up- and downgoing quantities in the time-reversed state. Recently, Wapenaar and Fokkema (2005) have suggested that when, as in our case, the medium outside the surrounding surface is homogeneous, the two terms under the integral in equation (11) are approximately equal, but opposite sign. In fact, this may be observed in Figure 3, by comparing panels (e) and (f).

Figure 7(a) shows the integrand of equation (10) for the $G_{11}(x_2, x_1)$ component of Green’s tensor between the two points of interest $x_1$ and $x_2$ in Figure 5. Note how the strongly scattered coda already identified in Figure 6 affects both negative and positive time-lags in the cross correlation. In Figure 7(b), the Green function $G_{11}(x_2, x_1)$ resulting from direct summation of the cross-correlation traces in panel (a) along the horizontal direction (blue) is compared to a reference solution (green). Note the emergence of the time-symmetry (across $t = 0$ s) from the non-symmetric cross-correlations. The intercorrelation Green function is time-symmetric instead of negative time-symmetric, as predicted by equation (10), because particle velocity Green functions were used in the example instead of particle displacement Green functions as in the derivations.

In Figure 8, the four components of the particle displacement Green tensor computed using the new method (in blue) [equation (10)] are compared to a directly computed reference solution (in green). The $G_{11}(x_2, x_1)$ component in panel (a) was already shown in Figure 7(b). Note the good match between the directly computed reference solutions and the Green functions.
computed using the new method, even at late times. The waveforms have been scaled and clipped to show the match in more detail. Some numerical noise at acausal time-lags (i.e., before arrival of the direct wave) can clearly be seen. This noise is probably due to a slight undersampling of the shear wavefield as the computational parameters have been set rather tightly to minimize computational cost. Note how the different source radiation patterns are reproduced accurately by the new modelling method; panels (a) and (b) show more P-wave energy (e.g., the first significant arrival), which is consistent with a point force source in the horizontal direction and the second point of interest at the same depth level, whereas panels (c) and (d) show more S-wave energy because of the maximum in S-wave radiation in the horizontal direction by a point force excitation in the vertical direction.

The time-series in Figure 7 bear little resemblance to the final Green function in Figure 8. Equation (10) sums signals such as those in Figure 7 along the horizontal axis and hence relies on the delicate constructive and destructive interference of time-reversed waves back-propagating through the medium, recombining and undoing the scattering at every discontinuity to produce the Green function. In Figure 7, each column represents the set of all waves travelling from point $x_1$ to a particular source location on the boundary, correlated with the Green functions from sources at that boundary source to $x_2$. Some of the energy travelling from $x_1$ to this boundary source may pass through $x_2$ before being recorded and therefore has part of its path in common with waves emitted from $x_2$ in the same direction (or wavenumber). The traveltimes associated with such identical parts of the path are eliminated in the cross correlation and the remaining traveltimes corresponds to an event in the Green function from $x_1$ to $x_2$. Similarly, some waves emitted from $x_2$ may travel to the boundary source location via $x_1$ and have a common section of path between $x_1$ and the boundary source. Again traveltimes on the common section will be eliminated and give rise to the same event in the Green function from $x_1$ to $x_2$ but at negative (acausal) times. Note that the vector wavenumbers involved for positive and negative times in general are not parallel since they are related to propagation of energy to the boundary through the background structure of the whole model (hence, one or other may not even exist for the same boundary source). Hence, waves at positive and negative times are reconstructed differently, even though the final Green function constructed is identical. All cross-correlations involving energy from $x_1$ that does not pass through $x_2$ or vice-versa are eliminated by destructive interference by summation of the columns. This process of constructive and destructive interference is discussed in detail by Snieder (2004) and Snieder et al. (2006) using the method of stationary phase.
DISCUSSION

We now discuss the computational aspects of the new modelling method and the implications for modelling and inversion. First, an estimate of the number of floating point operations is derived for both the initial and intercorrelation phase and compared to the cost of a sequence of conventional finite-difference computations. Then memory and storage implications are highlighted. This is followed by discussion on possibilities to speed up the initial modelling phase either by reducing the number of sources on the boundary or through the use of simultaneous sources. The limits of encoding and decoding using pseudo-noise sequences are also discussed. In Table I, parameters and variables mentioned in the computational discussion are summarized.

Floating point operations

Although the new methodology certainly offers unprecedented flexibility in the modelling of Green’s functions, since it allows the modeller to compute components that are common to all Green’s functions without first deciding on a particular source-receiver geometry, the computational cost of “looking up” a Green function by cross-correlation and summation of point of interest gathers cannot simply be ignored. For each Green’s function that is “looked up”, at least $N_S$ cross-correlations and summations have to be computed (where $N_S$ is the number of source locations on the boundary) and often more, depending on the particular type of wave equation (e.g., scalar or vector, acoustic, electromagnetic or elastodynamic) and the number of components (i.e., the spatial dimensionality of the problem). Thus, the computational cost of the new method, in number of floating point operations, should be expressed as:

$$C_{NEW} = C_{INIT} + N_{GF}C_{GREEN},$$

(12)

where $C_{INIT}$ and $C_{GREEN}$ are the cost of the initial conventional modelling phase and of looking up a single Green function (both specified below) and $N_{GF}$ is the number of Green functions that is looked up using the new method.

On the other hand, in the second, intercorrelation phase of the new method, the strict spatio-temporal sampling requirements of a typical full waveform modelling method (as governed by numerical accuracy and the Courant criterion) can be relaxed to Nyquist criteria. For example, in Appendix C it is shown that for a typical acoustic 2D finite-difference code with 2nd order accuracy in time and 4th order accuracy in space [denoted FD(2,4)], the ratio $b$ between the
time step $\Delta T'$ in the intercorrelation phase and the time step $\Delta T$ in the conventional modelling phase is:

$$b \equiv \frac{\Delta T'}{\Delta T} = \frac{N_T}{N_T'} \approx \frac{4 \left( \frac{c_{\text{max}}}{c_{\text{min}}} \right)}{\gamma},$$

(13)

where $c_{\text{max}}$ and $c_{\text{min}}$ are the maximum and minimum velocity in the medium, respectively and $\gamma$ is a constant relating to the specific FD code. For the FD(2,4) code $\gamma = 0.45$. $N_T$ and $N_{T'}$ are the number of time samples in the initial and intercorrelation phase respectively. Taking the maximum frequency $f_{\text{max}}$ in the final seismogram $f_{\text{max}} = 75$ Hz, $c_{\text{min}} = 1500$ m/s and $c_{\text{max}} = 4500$ m/s (salt) gives the ratio $b = 26.67$. Thus, the cost of looking up a Green function in the intercorrelation phase is substantially reduced by abandoning the oversampling.

In addition, to further reduce the cost of looking up Green’s functions, waveforms modelled in the initial phase are stored in the frequency domain in anticipation of the cross correlations in the intercorrelation phase (multiplication with the complex conjugate). This avoids having to recompute the Fourier transform of point of interest gathers when computing several Green functions involving the same point of interest.

In Appendix C it is shown that, based on the relaxed sampling requirements and use of the fast Fourier transform (FFT) to compute cross-correlations as outlined above, the total number of floating point operation (flops) required for the look-up of one component of a single Green tensor $C_{\text{GREEN}}$ is:

$$C_{\text{GREEN}} \sim (8c + 2)N_sN_{T'},$$

(14)

where $c$ is the number of cross-correlations that need to be computed for a single boundary source location when one component of Green’s tensor is looked-up. Hence, $c$ depends on the the type of wave equation (i.e., scalar or vector) and the spatial dimensionality of the problem, $n$. In Appendix C, $c$ is computed for the cases of two- and three-dimensional, acoustic and elastic modelling. The results are summarized in Table II. In equation (14) the cost of Fourier transforming the intercorrelation Green function back to the time-domain has been neglected as explained in Appendix C.

Note that using the new method, the simulation time $T$ has to be longer than in a conventional FD simulation. This can be understood by considering the reciprocal experiment in which one of two points of interest acts as a source: any energy that we want to time-reverse has to be recorded on the surrounding boundary. This means, e.g., that a particular (multiply scattered) wave in the Green function between the two points has to have had enough time to reach the boundary from that second point in order to be time-reversed. In general the
initial FD simulations need to run long enough such that most energy has left the grid. In the following we assume that this doubles the simulation time for the new method.

The computational cost of both the initial modelling phase and a sequence of conventional finite-difference simulations is proportional to $C_{FD}$, the number of flops required for a single conventional FD simulation. In turn, $C_{FD}$ is proportional to the number of gridpoints in each of $n$ dimensions, $N_X$, timesteps $N_T$ and the number of flops, $a$, required for the evaluation of the discrete temporal and spatial derivatives [e.g., $a = 22$ for a typical acoustic 2D FD code (see Table II)]. Thus we have:

$$C_{FD} \sim a N_T N_X^n.$$ (15)

Based on the above considerations, the number of floating point operations for a sequence of $N_M$ conventional FD calculations and for the new method now can be compared as [Appendix C, equation (36)]:

$$C_{CONV} = \frac{1}{2} m N_M C_{FD}$$ (16)

$$C_{NEW} = m N_M C_{FD} + m^2 N_G F C_{GREEN},$$ (17)

where $m = 1$ in case of acoustic modelling and $m = n$ in case of elastic modelling and $N_M$ is the minimum of the number of source and receiver locations considered in the modelling.

**Case 1:** $N_G F = N_M$ The new method is particularly attractive in applications where Green’s functions are desired between a large number of points interior to a medium, but where there are no common source or receiver points. In such a case, a separate conventional FD simulation is required for each Green tensor and, hence, $N_G F = N_M$. In this case, equations (16) and (17) are a straight line through zero and a vertically offset straight line, as a function of $N_M$, respectively and the new method becomes more efficient beyond the intersection point of the two lines. No other existing method could offer full waveforms at comparable computational cost. The method also offers great flexibility where the exact interior points are not known in advance since Green’s functions can be computed on an “as needed” basis from Green’s functions between points on the surrounding surface and its interior. We have shown how the latter Green functions constitute a common component of all Green functions in the medium through equation (10).

**Case 2:** $N_G F \sim N_M^2$ When the number of sources and receivers are approximately equal and one is interested in computing Green’s functions for all possible source/receiver combinations, the number of Green functions to be looked up $N_G F$ is proportional to the square of
the number of source/receiver points $N_M$: $N_{GF} \sim N_M^2$ and the new method may actually be less efficient than a sequence of conventional FD simulations. In such a case, equations (16) and (17) are a straight line through zero and a vertically offset parabola, as a function of $N_M$, respectively and, at best, there may be a region where the new method is more computationally efficient.

**Memory and storage**

Assuming that a standard isotropic elastic FD method is used (e.g., not relying on domain decomposition), the amount of run-time memory required for storage of the $(n/2)(n+3)$ field quantities (e.g., $v_i$ and $\sigma_{ij}$) and 3 medium parameters (e.g., $\rho$, $\lambda$ and $\mu$) in run-time memory requires at least $4[(n/2)(n+3)+3]N_X^6$ bytes (for a heterogeneous medium and calculations carried out in single precision). We note that for a medium size of $N_X = 1000$, a 3D elastic problem will require on the order of 45 Gbytes of primary memory. This number grows considerably for even more complex media (e.g., anisotropic), and the computations therefore typically rely on large shared memory machines or heavily parallelized algorithms running on clusters with high-performance connections. Using our methodology we compute a table of all point of interest gathers using high-end computational resources. The computations in the intercorrelation phase, on the other hand, would be performed on much smaller machines as they require a substantially smaller amount of primary memory and because they require only a subset of the inter-correlation table to be exported. We have shown how the point of interest gathers with Green functions constitute a common component of all Green functions in the medium through equation (10).

**Simultaneous sources: limits of encoding and decoding**

We also investigated exciting the boundary sources simultaneously by encoding the source signals using pseudo-noise sequences (Fan and Darnell, 2003) and with simultaneous sources distributed randomly in the medium (Derode et al., 2003) as two alternative ways to reduce the number of sources, and hence, the computational cost of the initial forward modelling phase. Such approaches have been investigated in attempts to speed-up conventional finite-difference simulations, although in surprisingly few published studies. Recent experimental evidence in passive imaging however, using techniques based on interferometry and time-reversal, seems to suggest that such an approach would be highly feasible for the new modelling method. For
instance, Wapenaar and Fokkema (2004) and Derode et al. (2003) show that, when the sources surrounding and inside the medium consist of uncorrelated noise sequences, their autocorrelation tends to a delta function and terms involving crosscorrelations between the different noise sequences can be ignored. However, it is well known in the field of communications analysis that Welch’s bound [Welch (1974)] poses a fundamental limit to the quality of separation of such pseudo-noise sequences of a given length, when emitted simultaneously. In Appendix D it is shown that, when making no assumptions about the Green function, the signal-to-interference ratio in the final modelled seismogram is proportional to $\sim \sqrt{N}$, where $N$ is the length of the sequences. Thus, the signal-to-interference ratio only improves as the square-root of the sequence length. Note that the number of sequences required, the so-called family size $M$ (equal to the number of boundary sources: $M = N_S$), does not influence the signal-to-interference ratio. A similar expression was recently derived by Snieder (2004) using a statistical approach to explain the emergence of the ballistic (direct wave) Green function through an ensemble of scatterers with uncorrelated positions. 

Although in principle, and in real-life experiments, it is possible to reduce such interference by time/event averaging, where data is “modelled” for free and all we have to do is listen longer (Snieder, 2004), in synthetic modelling of Green’s functions it is exactly the modelling itself that is expensive and therefore the use of pseudo-noise sequences for the purpose of interferometric, simultaneous source FD modelling is probably limited. In all explored cases, the limits of separation caused relatively high noise levels compared to the equivalent FD effort using the direct method described above.

**High-order approximations**

Whereas traditional approximate modelling methods typically impose restrictions with respect to the degree of heterogeneity in the medium of propagation or neglect high-order scattering, the new time-reversal modelling methodology allows us instead to compromise on noise level while maintaining high-order scattering and full heterogeneity in the medium. Recent experimental and theoretical work indicates that time-reversed imaging is robust with respect to perturbations in the boundary conditions (Derode et al., 2003; Snieder and Scales, 1998). For cases where the wave propagation is heavily dominated by multiple scattering even a single source may be sufficient to excite all wavenumbers in the model, and hence to refocus essential parts of a time-reversed signal (Draeger and Fink, 1999). Even when not all wavenumbers are
excited by a single source, such as in the examples above, it may be possible to substantially reduce the number of sources and still recover essential parts of the signal. van Manen et al. (2005) showed that even for as few as one sixteenth of the original number of sources they were able to reproduce amplitude and phase of an arrival of interest fairly accurately, but with an increased noise level. Clearly, the required number of sources will depend on the application. For many applications, the possibility to trade-off signal-to-noise ratio to CPU time without compromising on medium complexity or high-order scattering will be another attractive property of the new method.

**Synergies with inverse theory**

We anticipate that the new methodology will also have a significant impact on inversion since in some related areas relevant theory has already been developed. For example, Oristaglio (1989) has shown that the Porter-Bojarski integral [equation (9)] forms the basis for an inverse scattering formula that uses all the data. He proved that a three step imaging procedure, consisting of backpropagation (or focusing) of receiver and source arrays and temporal filtering, gives the scattering potential within the Born approximation. His formula also relies on complete illumination of a (three-dimensional) scattering object from a surface surrounding the object as our modeling method does. Rose (2002) argues that focusing, combined with time-reversal is the physical basis of exact inverse scattering and derives the Newton-Marchenko equation from these two principles.

In cases where the medium is relatively well known, but where the objective is to track some kind of non-stationary source within the volume, again the flexibility of the new method is a significant advantage, especially if the receivers are also non-stationary.

The new method also provides a flexible way to compute spatial derivatives of the intercorrelation Green functions with respect to both source and receiver coordinates for any region in the model, provided the points of interest are spaced closely enough in the initial modeling phase (as mentioned above, the spacing is only subject to storage limitations and does not influence the computational cost). This means that it is straightforward to consider other types of sources, such as pure P- and S-wave sources and detectors that are only sensitive to P- or S-waves [see e.g., Wapenaar and Haimé (1990) and Robertsson and Curtis (2002)]. Alternatively, the new modelling method could have been formulated directly in terms of crosscorrelations of Green functions for pure P- and S-wave sources on the boundary and pure P- and S-wave receivers in the points of interest (Wapenaar, 2004).
Another interesting aspect of the method is that it provides exactly those Green functions required for direct evaluation of higher-order terms in the Neumann series solution to multiple scattering. To illustrate this, consider again the case of isotropic point scatterers embedded in a homogeneous background medium. The complex scattering coefficient of scatterer \( i \) is denoted \( A_i \). Snieder and Scales (1998) show that the total multiply scattered wavefield \( u(x) \) at location \( x \) due to the (unperturbed) incident wavefield \( u(0)(x) \) can be written as:

\[
u(x) = u(0)(x) + \sum_i G(x, x_i) A_i u(0)(x_i) + \sum_{i \neq j} G(x, x_i) A_i G(x_i, x_j) A_j u(0)(x_j) + \cdots , \tag{18}
\]

where \( G(x_i, x_j) \) denotes Green’s function in the background medium between (scatterer) locations \( x_j \) and \( x_i \). This is the Neumann series solution. The first term on the r.h.s., \( u(0)(x) \), corresponds to the incident wavefield measured at the receiver location (propagating in the background medium). The second term corresponds to the Born (single scattering) approximation. The third term is a double summation over all possible paths including two scatterers, where the same scatterer cannot be visited on consecutive scattering events. Note that all Green functions in equation (18) (between scatterers and between scatterers and the observation point) can be computed efficiently using the new method. For the example of a homogeneous background medium given here, the required Green functions are simply free-space Green functions and the computation is trivial. However, equation (18) is easily generalized to the case of small perturbations with respect to more complicated background media and although the summations become volume integrations, the form of the Neumann series solution stays the same. In such cases, computation of the Green functions in the background medium may not be trivial and the new method provides such Green functions efficiently and flexibly. Note that we do not even have to specify beforehand which regions of the model we want to perturb.

Thus, we have shown how recent insights into the relationship between Green’s theorem and time-reversal can be extended to modelling of wave propagation by invoking reciprocity. We expect that this may significantly change the way we approach modelling and inversion of the wave equation in future.

**APPENDIX A: THE ELASTODYNAMIC REPRESENTATION THEOREM**

Here, for completeness, we derive the elastodynamic reciprocity theorem. We closely follow Snieder (2002). The elastodynamic reciprocity theorem relates two states \((A)\) and \((B)\) that can
occur in the same medium independently:

\[ \rho \omega^2 u^{(A)}_i + \partial_j \left( c_{ijkl} \partial_k u^{(A)}_i \right) = -f^{(A)}_i, \]
\[ \rho \omega^2 u^{(B)}_i + \partial_j \left( c_{ijkl} \partial_k u^{(B)}_i \right) = -f^{(B)}_i, \]  
(19)

where \( u^{(A)} \) and \( u^{(B)} \) are particle displacements for states \((A)\) and \((B)\), generated by excitations \( f^{(A)} \) and \( f^{(B)} \) respectively. The representation theorem can be derived by multiplying the first equation by \( u^{(B)}_i \) and the second by \( u^{(A)}_i \) and subtracting the results and integrating over a volume \( V \):

\[ \int_V \left\{ u^{(B)}_i \partial_j \left( c_{ijkl} \partial_k u^{(A)}_i \right) - u^{(A)}_i \partial_j \left( c_{ijkl} \partial_k u^{(B)}_i \right) \right\} \, dV \\
= -\int_V \left\{ f^{(A)}_i u^{(B)}_i - f^{(B)}_i u^{(A)}_i \right\} \, dV. \]  
(20)

The first term in curly brackets on the left hand side can be identified as one part of the integrated product rule for differentiation:

\[ \int_V u^{(B)}_i \partial_j \left( c_{ijkl} \partial_k u^{(A)}_i \right) \, dV = \]
\[ \int_V \partial_j \left( u^{(B)}_i c_{ijkl} \partial_k u^{(A)}_i \right) \, dV - \int_V \left( \partial_j u^{(B)}_i \right) c_{ijkl} \partial_k u^{(A)}_i \, dV, \]  
(21)

and the second term can be interpreted similarly. Inserting these expressions and using Gauss’s theorem to convert volume integrals to surface integrals, we get:

\[ \int_S \left\{ u^{(B)}_i n_j c_{ijkl} \partial_k u^{(A)}_i - u^{(A)}_i n_j c_{ijkl} \partial_k u^{(B)}_i \right\} \, dS \\
- \int_V \left\{ \left( \partial_j u^{(B)}_i \right) c_{ijkl} \left( \partial_k u^{(A)}_i \right) - \left( \partial_j u^{(A)}_i \right) c_{ijkl} \left( \partial_k u^{(B)}_i \right) \right\} \, dV \\
= -\int_V \left\{ f^{(A)}_i u^{(B)}_i - f^{(B)}_i u^{(A)}_i \right\} \, dV. \]  
(22)

Note that the second term on the left hand side disappears because of the symmetry properties of the elasticity tensor: \( c_{ijkl} = c_{klji} \). Thus we have:

\[ \int_S \left\{ u^{(B)}_i n_j c_{ijkl} \partial_k u^{(A)}_i - u^{(A)}_i n_j c_{ijkl} \partial_k u^{(B)}_i \right\} \, dS \\
= -\int_V \left\{ f^{(A)}_i u^{(B)}_i - f^{(B)}_i u^{(A)}_i \right\} \, dV. \]  
(23)

Note that the reciprocity theorem can be derived from this expression by assuming homogeneous boundary conditions and identifying both states \((A)\) and \((B)\) with a mathematical or Green state [i.e., a state involving Green’s functions rather than measured quantities (Snieder, 2002; Wapenaar and Fokkema, 2004)]. In the main text, equation (23) is used to derive the representation theorem.
APPENDIX B: COMPUTATION OF THE GRADIENT BY SPATIAL FILTERING

It is well known that when the wave field on a boundary satisfies outgoing (i.e., radiation or absorbing) boundary conditions, the wave field and its gradient (or traction) are directly related. For example, Holvik and Amundsen (2005) derive the following expressions in the frequency-wavenumber \((\omega, k)\)-domain that relate the upgoing components of particle velocity \(V^{(up)}(k)\) of a plane wave propagating with horizontal slowness \(p = (k/\omega)\) to the upgoing traction \(T^{(up)}(k)\) across a horizontal array of receivers (sources):

\[
T^{(up)}(k) = L_{TV}(k)V^{(up)}(k),
\]

where the particle velocity and traction vector are defined as follows:

\[
V^{(up)} = \begin{pmatrix} V_1^{(up)} \\ V_2^{(up)} \\ V_3^{(up)} \end{pmatrix},
\]

\[
T^{(up)} = \begin{pmatrix} T_1^{(up)} \\ T_2^{(up)} \\ T_3^{(up)} \end{pmatrix},
\]

and \(T\) denotes transposed. The \(3 \times 3\) matrix \(L_{TV}(k)\) is derived as (Holvik and Amundsen, 2005):

\[
L_{TV} = \frac{\rho \omega}{k_\phi} \begin{pmatrix} k_{z,\alpha} - \frac{k_x^2}{k_\beta^2} (k_{z,\alpha} - k_{z,\beta}) & k_x k_{z,\beta} (k_{z,\alpha} - k_{z,\beta}) & k_x [1 - 2k_\beta^{-2}k_\phi] \\ k_x k_{z,\beta} (k_{z,\alpha} - k_{z,\beta}) & k_{z,\alpha} - \frac{k_y^2}{k_\beta^2} (k_{z,\alpha} - k_{z,\beta}) & k_y [1 - 2k_\beta^{-2}k_\phi] \\ -k_x [1 - 2k_\beta^{-2}k_\phi] & -k_y [1 - 2k_\beta^{-2}k_\phi] & k_{z,\beta} \end{pmatrix}
\]

In equation (27), \(k_x\) and \(k_y\) are the components of the wavenumber vector parallel to the array of receivers (sources), \(k_r = (k_x^2 + k_y^2)^{1/2}\) is the length of the wavenumber vector and \(k_{z,\alpha} = (k_{z,\alpha}^2 - k_z^2)^{1/2}\) and \(k_{z,\beta} = (k_{z,\beta}^2 - k_z^2)^{1/2}\) are the P- and S-wave slownesses perpendicular to the array of receivers (sources) respectively with \(k_{\alpha} = (\omega/\alpha)\) and \(k_{\beta} = (\omega/\alpha)\) the P- and S-wavenumbers. In addition, an auxiliary quantity \(k_\phi = k_x^2 + k_{z,\alpha} k_{z,\beta}\) has been defined.

Similarly, for acoustic waves propagating in a single direction across an array, the pressure \(P(k)\) and its gradient \(\nabla P(k)\) are related through:

\[
\frac{\partial P}{\partial n} \equiv \mathbf{n} \cdot \nabla P = ik_{z,\alpha} P,
\]

where \(\mathbf{n}\) is the normal to the array and \(i\) is the imaginary unit. Note that these relations depend on material properties and require that the medium is (locally) laterally homogeneous.

Thus, equations (24) and (28) allow us to calculate the outgoing traction or pressure gradient associated with the modelled particle velocity or pressure on the surface surrounding
the medium because absorbing boundaries were included right outside the enclosing boundary during the modelling.

The implementation of equations (24) and (28) is straightforward when the wave field is recorded (or emitted) on a linear array of regularly spaced receivers (sources) embedded in a homogeneous medium. In that case, the point of interest gathers can be directly transformed to the frequency-wavenumber domain and the matrix multiplication carried out explicitly before the components of the resulting traction vector inverse-Fourier transformed to the space-frequency domain.

Alternatively, when the medium is laterally varying or the array of receivers (sources) is curved, equation (24) can be implemented by designing spatially compact filters that approximate the terms of $L_{TV}$ (or, in the acoustic case, $i \omega q_{\alpha}$) and filtering the data in the space-frequency domain. Such an approach has been used in, for instance, the seabed seismic setting to decompose the wave field measured at the seabed into up- and downgoing P- and S-waves (Røsten et al., 2002; van Manen et al., 2004) and is based on solving a linear least-squares problem with (in-)equality constraints to find a small number of spatial filter coefficients with a wavenumber spectrum that best matches the spectrum of the analytical expression. Since the analytical expressions [equations (27) and (28)] are functions of frequency, this optimization is carried out for each frequency separately. The laterally varying seafloor properties are accommodated by designing such compact filters for the particular seafloor properties that are present at each receiver location. These filters are then applied to the point of interest gathers in the space-frequency domain by space-variant convolution.

Note that the filter coefficients only have to be optimized once for a particular model and can be reused for all Green’s functions that are computed in the intercorrelation phase. This approach was tested on acoustic data computed for the Pluto model (modelled with $\beta = 0 \text{ m/s}$) and gave good results.

**APPENDIX C: FLOATING POINT COUNT**

The FD gridspacing $\Delta X$ is determined by the shortest wavelength $\lambda_{\text{min}} = (c_{\text{min}}/f_{\text{max}})$ in the medium, where $c_{\text{min}}$ and $f_{\text{max}}$ are the minimum velocity in the medium and maximum frequency of interest respectively. The 2D FD code used in the example [2nd order accuracy in time, 4th order accuracy in space, denoted FD(2,4)] requires at least 5-8 gridpoints per
minimum wavelength to minimize so-called numerical dispersion [see e.g., (Levander, 1988) and (Robertsson et al., 1994)], thus:

$$\Delta X \leq \frac{\lambda_{\text{min}}}{8} = \frac{c_{\text{min}}}{8 f_{\text{max}}}.$$  

(29)

The time step $\Delta T$ in the FD simulation is determined by the Courant stability criterion:

$$\Delta T \leq \gamma \frac{\Delta X}{c_{\text{max}}} = \frac{\gamma}{8 f_{\text{max}}} \left( \frac{c_{\text{min}}}{c_{\text{max}}} \right),$$

(30)

where $\gamma$ is a constant relating to the specific FD code. For the FD(2,4) code $\gamma = 0.45$ (the maximum Courant number allowed in the specific FD code that we used is $\sim 75\%$ lower than what is normal for this type of code. This has to do with controlling stability of the specific absorbing boundary conditions used). The number of floating point operations (flops) required for a single simulation using the FD code is directly proportional to the number of gridpoints $N_X^n$ (where $n$ is the dimension of the medium) and timesteps $N_T$, which in turn are determined by $\Delta X$ and $\Delta T$ respectively:

$$C_{FD} \approx a N_T N_X^n.$$  

(31)

Here, $a$ denotes the number of flops required for the evaluation of the discrete temporal and spatial derivatives (e.g., $a = 22$ for a typical acoustic 2D FD code). Note that both the cost of conventional simulations for $N_M$ sources/receivers and the initial computations of the new method for $N_S$ boundary sources are proportional to $N_M C_{FD}$ and $N_S C_{FD}$ respectively.

The time step in the inter-correlation phase $\Delta T'$ on the other hand is governed by the Nyquist frequency and can be as large as $1/(2 f_{\text{max}})$. Thus, the ratio between the two time steps (or between the number of samples) in the seismogram is:

$$b \equiv \frac{\Delta T'}{\Delta T} = \frac{N_T}{N_{T'}} \approx 4 \left( \frac{c_{\text{max}}}{c_{\text{min}}} \right).$$

(32)

For example, taking $f_{\text{max}} = 75 Hz$, $c_{\text{min}} = 1500 m/s$ and $c_{\text{max}} = 4500 m/s$ (salt), the gridpacing, timestep and sampling ratio become: $\Delta X = 2.5 m$, $\Delta T = 2.5 e - 4 s$ and $b = 26.67$ respectively.

The waveforms computed in the initial phase are stored in the frequency domain using the FFT. The traces are padded with $N_{T'}$ zeros to avoid end effects. Transforming a waveform to the frequency domain therefore takes on the order of:

$$C_{FFT} \approx N_{T'} \log_2(N_{T'} + 1)$$

(33)

twice the number of time samples in the desired (one-sided) final seismogram and we have made use of the fact that the FFT of two real traces can be computed at once.
The cost of looking up a single component of Green’s tensor can now be computed as follows. The number of cross-correlations, \( c \), that need to be computed for a boundary source location depends on the type of wave equation (i.e., scalar or vector) and the spatial dimensionality of the problem, \( n \). For an acoustic problem there is only a single, scalar quantity and this does not increase with the spatial dimensionality. Therefore, \( c = 2 \), the number of terms in the integrand in equation (11). For an elastic problem, the implicit Einstein summation (for repeated indices) in equation (10) and the two terms in the integrand lead to \( c = 2n \). The acoustic and elastic cases can be written in the same form by defining the quantity \( m \), which equals the dimension of the medium \( n \) when considering the elastic wave equation and one when considering the acoustic wave equation:

\[
    m = \begin{cases} 
        1 & \text{(acoustic)} \\
        n & \text{(elastic)} 
    \end{cases}
\]

and thus \( c = 2m \). Complex multiplication of the positive frequencies for all source locations on the boundary and the \( c \) cross-correlations takes \( 6cN_SN_T' \) operations. Complex addition of the \( c \) cross-correlation gathers takes \( 2(c-1)N_SN_T' \) operations. Multiplying the cross-correlation gathers with the (varying) weighting factor along the boundary takes \( 2N_SN_T' \) operations. Summing the weighted cross-correlations for all source locations takes \( 2(N_S-1)N_T' \) flops. Thus, the total number of flops required for the inter-correlation of a single component of Green’s tensor is approximately:

\[
    C_{GREEN} \approx (8c + 2)N_SN_T' + C_{FFT},
\]

where \( C_{FFT} \) relates to the final inverse FFT. Note that \( C_{FFT} \) can typically be neglected since in most case of interest \((8c + 2)N_S \gg \log_2(N_T' + 1)\).

As pointed out above, the computations of the initial phase are approximately proportional to \( C_{FD} \), the cost of a single, conventional finite-difference simulation. For the new modelling method, however, even to compute a single component of Green’s tensor, we need conventional forward simulations for point sources in all \( n \) orthogonal directions (note that this is only a real disadvantage compared to a conventional modelling method when we are not interest in all components of Green’s tensor). In addition we need to compute the FFT of a zero padded trace for all \( N \) points of interest. Again making use of the fact that the waveforms are real-valued, the total number of flops for the initial phase thus becomes:

\[
    C_{INIT} = mN_S(C_{FD} + NC_{FFT}).
\]
Note that the magnitude of the second term in the bracket in the expression above typically is negligible compared to the first term and therefore is neglected in the following.

We are now in the position to compare the cost of the new method $C_{\text{NEW}}$ to a sequence of $N_M$ conventional finite-difference simulations $C_{\text{CONV}}$. Note, as explained in the main text, that the conventional simulations only run for half the simulation time $T$. Thus, using equation (35), we arrive at:

$$C_{\text{CONV}} = \frac{1}{2} mN_M C_{FD}$$

$$C_{\text{NEW}} = mN_M C_{FD} + m^2 N_{GF} C_{\text{GREEN}},$$

where we have assumed that we are interested in all $n^2$ components of Green’s tensor.

Note that in the above discussion, we have ignored the cost of modelling the response to the second source type (i.e., the dipole or deformation rate tensor sources). As explained in detail in Appendix B, the gradient (or traction) can be computed from the pressure (or particle displacement) recordings through a spatial filtering procedure. The cost of this type of spatial filtering is typically insignificant compared to the FD simulations. In addition we note that Wapenaar and Fokkema (2005) have shown for the acoustic case that, when the surrounding surface is large enough radius such that Fraunhofer far-field conditions apply, only monopole sources are required to compute (time-integrated) Green functions.

**APPENDIX D: THE WELCH BOUND AND LIMITS TO ENCODING USING PSEUDO-NOISE SEQUENCES**

In communications analysis, the problem of encoding and decoding signals using pseudo-noise sequences and its limits are well known. In particular, Welch has shown (Welch, 1974) that for any family of $M$ (unit energy) sequences $\{a_n^{(i)}\}, i = 0, \ldots, M-1, n = 0, \ldots, N-1$ of length $N$, a lower bound on the maximum (aperiodic) cross correlation or off-peak autocorrelation is:

$$C_{\text{max}} = \max\{C_{am}, C_{cm}\} \geq \sqrt{\frac{M - 1}{M(2N - 1) - 1}}$$

(38)
where $C_{am}$ and $C_{cm}$ are the maximum off-peak autocorrelation and maximum cross correlation values defined by:

\[
C_{am} = \max_i \max_{1<\tau\leq N-1} |C_{i,i}(\tau)|
\]

\[
C_{cm} = \max_{i\neq j} \max_{0<\tau\leq N-1} |C_{i,j}(\tau)|
\]

and $C_{i,j}$ is the discrete aperiodic correlation function of the sequences $a_n^{(i)}$ and $a_n^{(j)}$ defined as:

\[
C_{i,j}(\tau) = \begin{cases} 
\sum_{n=0}^{N-1-\tau} a_n^{(i)} a_{n+\tau}^{(j)}, & 0 \leq \tau \leq N - 1 \\
\sum_{n=0}^{N-1+\tau} a_n^{(i)} a_{n-\tau}^{(j)}, & -N + 1 \leq \tau < 0 \\
0, & |\tau| \geq N.
\end{cases}
\]

Note that the Welch bound [equation (38)] holds without reference to a particular type of sequence set [e.g., maximal, Kasami or Gold sequences (Fan and Darnell, 2003)]. This means that when we encode signals using sequences of any such family, superpose the encoded signals and subsequently decode using cross-correlation, there will be some point in the decoded output where the interference between the original data sequences is at least $C_{max}$. We can estimate the best possible performance that can be expected [without making any (questionable) assumptions about the uncorrelatedness of the Green functions from boundary sources to points of interest] by looking at the (rms) expected signal-to-interference ratio when all members $a_n^{(i)}$ of the sequence set are simply added and the result $r_n$ is autocorrelated. Thus we have:

\[
r_n = \sum_i a_n^{(i)}, \tag{39}
\]

and the corresponding autocorrelation $C_r$,

\[
C_r(\tau) = \sum_{n=0}^{N-1-\tau} r_n r_{n+\tau}, \quad 0 \leq \tau \leq N - 1, \tag{40}
\]

which, using equation (39) can be written:

\[
C_r(\tau) = \sum_i \sum_{n=0}^{N-1-\tau} a_n^{(i)} a_{n+\tau}^{(i)} + \sum_{i\neq j} \sum_{n=0}^{N-1-\tau} a_n^{(i)} a_{n+\tau}^{(j)}. \tag{41}
\]

The first term denotes the diagonal, or signal term $C_D(\tau)$ whereas the second term $C_C(\tau)$ denotes the cross-terms related to the (unwanted) interference between the different codes and contains a double summation.
Equation (41) mimics the structure of the interferometric modelling equations [equations 10 and 11]: when the boundary source signals are encoded using pseudo-noise sequences and excited simultaneously, it is their superposition (convolved with the Green function) that is recorded in the points of interest, and when calculating the Green function, decoding, cross-correlation and summation are implicit in a direct correlation (just like in a typical retrieval of the Green function from uncorrelated noise sources). Thus, as long as we assume that the Green functions from the boundary to the points of interest do not influence the signal-to-interference ratio, an estimate of the ratio can be found by analyzing equation (41). This is done by comparing the expected magnitude of the first term to the magnitude of the second term. Since the diagonal term consists of a sum of the autocorrelations of the sequences, its magnitude is maximum at zero-lag ($\tau = 0$) and equal to the family size $M$:

$$C_D(0) = \sum_i \sum_{n=0}^{N-1} a_n^{(i)} a_n^{(i)} = M$$

(42)
because the signals are unit energy. The expected value of the second term, is actually calculated by Welch as part of his derivation of equation (38). In fact, Welch’s original statement is basically a lower bound on the root-mean-square value of a family unit energy signals:

$$C_{\text{rms}} \geq \sqrt{\frac{M - 1}{M(2N - 1) - 1}} (43)$$

and since $C_{\text{max}} \geq C_{\text{rms}}$, equation (38) follows. Thus, the Welch bound gives the rms-value of each of the terms within the double sum in $C_C(\tau)$. Note that the sign of each of these $M(M-1)$ terms is not directly specified through the Welch bound, the only thing we can say about the sign is that its expected value is zero when the DC component of the sequences vanishes and the sequences are (in the ensemble average) uncorrelated. Thus, we estimate the magnitude of the term $C_C(\tau)$ by calculating the variance of:

$$C_C(\tau) = \pm C_{\text{rms}} \pm C_{\text{rms}} \pm C_{\text{rms}} \ldots$$

(44)
which simply is:

$$\langle C_C(\tau) \rangle \frac{1}{2} = \sqrt{M(M - 1)C_{\text{rms}}}$$

(45)
and the ratio of the signal term to the interference term becomes:

$$\frac{C_D(0)}{\langle C_C(\tau) \rangle \frac{1}{2}} \approx M \left( \frac{M(2N - 1) - 1}{M(M - 1)^2} \right)$$

(46)
When both the sequence length $N$ and the family size $M$ are much larger than one this becomes:

$$\frac{C_D(0)}{\langle C_C(\tau) \rangle^2} \approx \sqrt{2N}$$

(47)

Thus, the signal-to-interference ratio improves as the square-root of the sequence length.

REFERENCES


Fig. 1: Flowchart of the proposed modelling method. The method consists of two main phases: An initial phase (top) that creates a partial modelling solution that is common to all Green’s functions, computed only once using a conventional forward modelling algorithm. Followed by a second, inter-correlation phase (bottom) where desired Green functions are computed from the partial modelling solution using only cross-correlation and summation, without the need for additional conventional modelling.

Fig. 2: 2D acoustic model and snapshot of a boundary source wavefield: Three isotropic point scatterers (large black dots) embedded in a homogeneous background medium $c_0 = 750$ m/s of infinite extent. Stars (*) mark every second source location on a surface enclosing the medium. Small dots (.) mark possible source/receiver locations (so-called points of interest) for Green function inter-correlation. Triangles mark one of many (cross-well) source/receiver configurations that can be evaluated using the new method. In the initial phase, the wavefield is computed for all boundary sources separately and stored in all points of interest. A source wavefield is shown for one of the boundary source locations.

Fig. 3: Modelled waveforms for all boundary sources in two points of interest and their cross-correlation: (a) Monopole response in point [-50,0] and (b) corresponding dipole response computed by spatial filtering [see text for details]. (c) Dipole response in point [50,-50] computed by spatial filtering and (d) corresponding monopole response. (e) Cross correlation of (a) and (c). (f) Cross correlation of (b) and (d). The difference between gathers (e) and (f), weighted by $\rho^{-1}$, forms the integrand of equation (11). (g) Intercorrelation Green function (green) and a directly computed reference solution (blue). Insets show details of the signals in time-intervals bounded by dashed boxes. Note the anti-symmetry of the intercorrelation Green function across $t = 0$ s, as predicted by equation (11). The waveforms have been slightly offset vertically from each other to better illustrate the match.

Fig. 4: Comparison of Green’s functions computed with the new interferometric modeling method and a reference solution. (a) reference, (b) new, and (c) difference ($\times 10$) between the Green functions for the cross-well transmission setting shown in the inset. (d) reference, (e) new, and (f) difference ($\times 10$) for the cross-well reflection setting shown in the inset.

Fig. 5: P-wave velocity of a 2D elastic marine seismic model. The color scale is clipped to display weak velocity contrasts (P-wave velocity of salt is 4500 m/s).
Fig. 6: Point-of-interest gather for the left point in Figure 5, showing the horizontal component of particle velocity in the point of interest due to individual point force sources in the horizontal direction on the boundary. This is one of four required particle velocity Green’s function gathers computed in the initial phase, needed in the construction of all Green’s functions involving that point.

Fig. 7: Green’s function inter-correlation gather (weighted) for the two points shown in Figure 5. The low correlation amplitude for boundary sources 620-800 corresponds to the shadow of the salt body. (b) Green’s function computed by direct summation of the cross-correlation traces in panel (a) along the horizontal direction (blue) compared to a reference solution (green). Note the good match and the emergence of the time-symmetry from the non-symmetric cross-correlations.

Fig. 8: Components of Green’s tensor $G(x_2, x_1)$ computed by summation of weighted intercorrelation gathers using the new method (blue) compared to reference solutions computed using a conventional FD method (green). (a) $G_{11}(x_2, x_1)$, (b) $G_{12}(x_2, x_1)$, (c) $G_{21}(x_2, x_1)$, (d) $G_{22}(x_2, x_1)$. For details see text.
FIGURES
FIG. 1 Flowchart of the proposed modelling method. The method consists of two main phases: An initial phase (top) that creates a partial modelling solution that is common to all Green’s functions, computed only once using a conventional forward modelling algorithm. Followed by a second, inter-correlation phase (bottom) where desired Green functions are computed from the partial modelling solution using only cross-correlation and summation, without the need for additional conventional modelling.
FIG. 2 2D acoustic model and snapshot of a boundary source wavefield: Three isotropic point scatterers (large black dots) embedded in a homogeneous background medium $c_0 = 750$ m/s of infinite extent. Stars (*) mark every second source location on a surface enclosing the medium. Small dots (.) mark possible source/receiver locations (so-called points of interest) for Green function inter-correlation. Triangles mark one of many (cross-well) source/receiver configurations that can be evaluated using the new method. In the initial phase, the wavefield is computed for all boundary sources separately and stored in all points of interest. A source wavefield is shown for one of the boundary source locations.
FIG. 3 Modelled waveforms for all boundary sources in two points of interest and their cross-correlation. (see captions for full text)
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FIG. 7  (a) Green’s function inter-correlation gather and summation. (see captions for full text)
FIG. 8 Components of Green’s tensor $G(x_2, x_1)$ computed by summation of weighted intercorrelation gathers using the new method (blue) compared to reference solutions computed using a conventional FD method (green). (a) $G_{11}(x_2, x_1)$, (b) $G_{12}(x_2, x_1)$, (c) $G_{21}(x_2, x_1)$, (d) $G_{22}(x_2, x_1)$. For details see text.
Tables
<table>
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<th>parameter</th>
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<td>$f_{\text{max}}$</td>
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<tr>
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**TABLE I** Description of all parameters and variables mentioned in the computational discussion
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TABLE II Values for the different parameters in 2D and 3D acoustic and elastic modelling