Elastic Autofocusing via Single-Sided Marchenko Inverse Scattering
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SUMMARY
Recently it has been shown that acoustic Green’s functions from sources inside of a medium can be recovered using single-sided reflection data by iteratively solving a Marchenko equation. In recent work, we conjectured how this scheme can be extended in the case of elastic media. In this work, we present a mathematical derivation that validates our previous conjecture, and exemplify it with a numerical example in a solid elastic medium.

INTRODUCTION
A novel technique known as autofocusing or Marchenko redatuming has been introduced recently to construct the acoustic Green’s function from a virtual source inside of the subsurface (Broggini et al., 2012; Wapenaar et al., 2012, 2013). Autofocusing forgoes the need for sources or receivers at the position of the virtual source, and only requires single-sided data consisting of the reflection data and an estimate of the direct wave from the virtual source to the receivers. The reflection data may be acquired through standard surface acquisition, while the direct arrival can be computed using a smooth estimate of the subsurface velocities. Acoustic autofocusing has been validated theoretically and numerically, and comprises a data-driven method to recover true-amplitude internal reflections from a virtual source inside of the medium.

Recent work extended autofocusing to solid media governed by elastodynamic wave equations (da Costa et al., 2014). We conjectured an elastic scheme and showed how an approximation of that scheme yields good results. Here, we prove our conjecture through the use of one-way elastic reciprocity theorems, and illustrate the method with a numerical example.

GREEN’S FUNCTIONS
We are interested in recovering the components of the elastodynamic Green’s function $G$ that satisfies the following wave equation

$$
\partial_t G_{ij,q}^{(t,f)}(x,x',\omega) - i\omega \rho(x) G_{ij,q}^{(t,f)}(x,x',\omega) = -\delta_{iq} \frac{\partial}{\partial x_j} \delta(x-x')
$$

Here, $\rho(x)$ is the density. Green’s function’s superscripts represent the measured quantity and source type, and subscripts refer to the receiver and source components. Its arguments are the receiver position, source position, and frequency. For instance, $G_{ij,q}^{(t,f)}(x,x',\omega)$ is the $ij$ component of the stress measured at $x$ that was caused by a $q$-directional force density source placed at $x'$. The elastic medium is assumed to be lossless, homogenous and nonreflecting above the source depth ($z < z_0$), but heterogeneous and arbitrarily complex below the source (Fig. 1). The boundary $\partial\mathcal{D}_0$ between the homogenous and heterogeneous regions is transparent. On the boundary, down- (+) and up-going (−) components of the stresses are given by

$$
G^{+}_{ij,q}(x_0,x''_0,\omega)|_{x_0 \in \partial\mathcal{D}_0} = -\frac{1}{2} \delta_{ij} \delta(x_0 - x''_0) \quad (1)
$$

$$
G^{-}_{ij,q}(x_0,x''_0,\omega)|_{x_0 \in \partial\mathcal{D}_0} = \frac{1}{2} G^{+}_{ij,q}(x_0,x'_0) \quad (2)
$$

FOCUSING FUNCTIONS
In the derivation of the Marchenko inverse scattering equation for 1D and 3D acoustic media, certain ‘fundamental’ solutions of their respective wave equations play an important role. In these cases, the fundamental solution is a noncausal wavefield shaped in such a way that it focuses at a certain point at $t = 0$, then propagates as a diverging wave that is only up- or down-going. Here we define one of these focusing functions in the context of elastic media through the following condition

$$
F^{+}_{iz,q}(x_m,x'_m,\omega) = -\frac{1}{2} \delta_{iq} \delta(x_m - x'_m) \quad (3)
$$

The focusing function $F$ is defined in a reference medium that, for $z < z_m$ is identical to the medium where $G$ is defined, but is nonreflecting and homogenous below $z_m$ (Fig. 2).

Figure 1: Model over which $G$ is defined.

Figure 2: Model over which $F$ is defined.
GREEN’S FUNCTION REPRESENTATIONS

Elastodynamic wavefields that share a region satisfy relations commonly referred to as reciprocity theorems. These wavefields also satisfy representation theorems when decomposed into up- and down-going components. We recall two reciprocity relationships in the frequency domain that relate two elastic states A and B that share the region $\mathcal{D}$ (Wapenaar and Berkhout, 1989). We take $\mathcal{D}$ to be the region comprised between boundaries $\partial D_0$ and $\partial D_m$, to obtain the following reciprocity theorems:

$$\int \{ v_i^B - v_i^A + \tau_i^B - \tau_i^A \} \, d^2 x = \int \{ v_i^B + v_i^A - \tau_i^B - \tau_i^A \} \, d^2 x$$

(4)

$$\int \{ b_i^B - b_i^A + \tau_i^B - \tau_i^A \} \, d^2 x = \int \{ b_i^B + b_i^A - \tau_i^B - \tau_i^A \} \, d^2 x$$

(5)

Here the asterisk (*) denotes complex conjugation, $v_i$ is the $i$th component of the particle velocity and $\tau_{ij}$ is the component $ij$ of the stress tensor.

Equation (4) can be simplified by noting that the terms $v_i^B v_i^A$ and $\tau_i^B \tau_i^A$ contribute the same energy to the left-hand side of the equation, while the terms $v_i^B \tau_i^A$ and $\tau_i^B v_i^A$ contribute the same energy to the right-hand side (Wapenaar and Berkhout, 1989). This yields the following relationship:

$$\int \{ v_i^B - v_i^A - \tau_i^B - \tau_i^A \} \, d^2 x = \int \{ v_i^B + v_i^A - \tau_i^B - \tau_i^A \} \, d^2 x$$

(6)

We substitute the quantities of state A for those of the focusing function $F$ and observe that $F$ has no up-going field at $\partial D_m$. State B is taken to be the one corresponding to the Green’s function $G$. By applying the focusing conditions of equations (1) and (3), equation (6) is reduced to:

$$- G^{(v,f)}(x', \omega) = - F^{(v,f)}(x', \omega) + \int \{ G^{(v,f)}(x, \omega) F^{(v,f)}(x', \omega) - G^{(v,f)}(x, \omega) F^{(v,f)}(x', \omega) \} \, d^2 x.$$  

(7)

Equation (5) relating time-reversed focusing function to the Green’s function can be developed analogously to yield:

$$- G^{(v,f)}(x'', \omega) = - F^{(v,f)}(x'', \omega) + \int \{ G^{(v,f)}(x, \omega) F^{(v,f)}(x'', \omega) - G^{(v,f)}(x, \omega) F^{(v,f)}(x'', \omega) \} \, d^2 x.$$  

(8)

After summing equations (7) and (8) and applying the elastodynamic reciprocity theorem, we obtain relationships between the velocity and stress components of the Green’s function and those of the focusing function.

$$- G^{(v,f)}(x''', \omega) = H^{(v,f)}(x'', \omega) - \int \{ G^{(v,f)}(x, \omega) H^{(v,f)}(x'', \omega) - G^{(v,f)}(x, \omega) H^{(v,f)}(x'', \omega) \} \, d^2 x.$$  

(9)

$$- G^{(v,f)}(x''', \omega) = H^{(v,f)}(x'', \omega) - \int \{ G^{(v,f)}(x, \omega) H^{(v,f)}(x'', \omega) - G^{(v,f)}(x, \omega) H^{(v,f)}(x'', \omega) \} \, d^2 x.$$  

(10)

where $H^{(v,f)}(x, \omega) = F^{(v,f)}(x, \omega) - F^{(v,f)}(x, \omega)$; the · stands for $v$ or $\tau$ components in the superscript, and $i$ or $iz$ accordingly in the subscript.

By substituting equation (2) in equations (9) and (10), these can be written in terms of the reflectivity instead of up-going Green’s functions. Furthermore, by applying the appropriate wavefield decomposition operator (Wapenaar and Berkhout, 1989), the $p$-directional force density sources $f$ can be turned into $N$-potential sources (where $N$ is P, SH, or SV). Defining

$$G^{(\phi)}(x, \omega) = \left[ G^{(\phi,v)}(x, \omega), G^{(\phi,\tau)}(x, \omega), G^{(\phi,\omega)}(x, \omega), G^{(\phi,x)}(x, \omega), G^{(\phi,y)}(x, \omega), G^{(\phi,z)}(x, \omega) \right],$$

$$H^{(\phi)}(x, \omega) = \left[ H^{(\phi,v)}(x, \omega), H^{(\phi,\tau)}(x, \omega), H^{(\phi,\omega)}(x, \omega), H^{(\phi,x)}(x, \omega), H^{(\phi,y)}(x, \omega), H^{(\phi,z)}(x, \omega) \right],$$

and

$$R = \left[ \begin{array}{cccccc} R^{(v,v)}(x, \omega) & R^{(v,\tau)}(x, \omega) & R^{(v,\omega)}(x, \omega) & R^{(v,x)}(x, \omega) & R^{(v,y)}(x, \omega) & R^{(v,z)}(x, \omega) \\ R^{(\tau,v)}(x, \omega) & R^{(\tau,\tau)}(x, \omega) & R^{(\tau,\omega)}(x, \omega) & R^{(\tau,x)}(x, \omega) & R^{(\tau,y)}(x, \omega) & R^{(\tau,z)}(x, \omega) \\ R^{(\omega,v)}(x, \omega) & R^{(\omega,\tau)}(x, \omega) & R^{(\omega,\omega)}(x, \omega) & R^{(\omega,x)}(x, \omega) & R^{(\omega,y)}(x, \omega) & R^{(\omega,z)}(x, \omega) \\ \end{array} \right],$$

they can be condensed into a time domain matrix equation:

$$G^{(\phi)}(x'', \omega, t) = H^{(\phi)}(x'', \omega, t) + \int \int_{\partial D_m} R(x', \omega) \, d^2 x.'
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It can be shown that $H$ is a focusing function as well as $F$. This means that equation (11) is an integral relationship between a focusing function $H$ and the reflectivity. Since the reflectivity is known, we only need to recover the focusing function $H$ in order to reconstruct $G$.

3D ELASTIC MARCHENKO EQUATION

In the 1D formulation, the corresponding $H$ is a fundamental solution that focuses at $x''_0$ and diverges towards $-\infty$. It is composed of a direct wave component and a coda that at $t = 0$ annihilates the scattering of the direct wave at the focusing point. A similar ansatz for the shape of $H$ was made in Wapenaar et al. (2013), where the direct wave was taken to be the reverse of a wave from the virtual source to the receivers, and the coda was assumed to arrive after the direct wave. Extending this ansatz for elastic media is nontrivial as elastic propagation exhibits different polarization of waves ($P$ and $S$) that travel at different speeds. In the general case, a direct wave from a force density at the virtual source location will generate $P$-, $SH$- and $SV$-wave arrivals, as well as their conversions. However, when a $N$-potential source is used, only $N$-waves and their conversions will propagate. Therefore, we will develop the focusing function $H$ in equation (11) separate Marchenko equations for each source potential type. Let us define $I^N_k(x''_0,x'_m)$ as the travel time of the first arrival at $x''_0$ of a wave generated by an $N$-potential source at $x'_m$. We make the assumption that $H^N_k(x''_0,x'_m,t)$ can be written as a sum of a direct wave and consequent coda:

$$H^N_k(x''_0,x'_m,t) = G^N_k(x''_0,x'_m,-t) + \theta(t + I^N_k(x''_0,x'_m)) \mathbf{M}^N_k(x''_0,x'_m,t).$$  \hspace{1cm} (12)

Here $G^N_k(x''_0,x'_m,t)$ is the first arrival $x''_0$ from an $N$-potential source at $x'_m$, $\theta$ is the Heaviside step function, and $\mathbf{M}^N_k(x''_0,x'_m,t)$ denotes the scattered coda. Applying this ansatz for the focusing function $H$ in equation (11), and evaluate it for times before the first arrival $I^N_k(x''_0,x'_m)$ we obtain the 3D elastic Marchenko equation:

$$0 = \int_{\partial B_0} \int_{-\infty}^{\infty} R(x''_0,x'_m,0,t) \left( G^N_k(x''_0,x'_m,-t) - \mathbf{M}^N_k(x''_0,x'_m,-t) \right) d\tau d^2x_0$$

$$+ \int_{\partial B_0} \int_{-\infty}^{\infty} R(x''_0,x'_m,0,t) \mathbf{M}^N_k(x''_0,x'_m,\tau) d\tau d^2x_0$$

$$+ \mathbf{M}^N_k(x''_0,x'_m,-t)$$  \hspace{1cm} (13)

3D ELASTIC AUTOFOCUSING

We define two fields $E^+_k(x_0,x_F,t)$ and $E^-_k(x_0,x_F,t)$ that will be iterated and combined to yield both the focusing function $H$ and the Green’s function $G$. For $k \geq 0$, we initialize $E^-_k = 0$ and define

$$E^+_k(x_0,x_F,t) = G^N_k(x_0,x_F,-t) - \theta(t + I^N_k(x''_0,x'_m)) E^+_{k-1}(x_0,x_F,-t)$$  \hspace{1cm} (14)

$$E^-_k(x_0,x_F,t) = \int_{\partial B_0} \int_{-\infty}^{\infty} R(x''_0,x_0,\tau) E^+_{k-1}(x_0,x_F,\tau) d\tau$$  \hspace{1cm} (15)

We call this scheme elastic autofocusing, given the similarity with previous autofocusing schemes of Rose (2001) and Wapenaar et al. (2012). When it converges, it can be shown from the Marchenko integral of equation (13) that

$$E^+(x''_0,x_F,t) = -\mathbf{M}^N_k(x''_0,x'_m,-t)$$  \hspace{1cm} (16)

and

$$E^-(x''_0,x_F,t) = H^N_k(x''_0,x'_m,-t).$$  \hspace{1cm} (17)

A direct consequence of these relationships is that the Green’s function can be recovered by combining the up-going field $E^-$ and the down-going field $E^+$. For $k \geq 0$:

$$\mathbf{G}^N_k(x''_0,x_F,t) = E^+(x''_0,x_F,t) + E^-(x''_0,x_F,t).$$  \hspace{1cm} (18)

We draw attention to equation (15) which is the exact elastic receiver side wavefield extrapolation in elastic RTM given by Ravasi and Curtis (2013). We also note that it requires velocity and stress recordings from force density and deformation rate density sources, which are not all readily available from field data. Though previous work has only used velocity components from force sources, we focus on the exact formulation.

NUMERICAL RESULTS

The 2D solid-Earth model used in the numerical experiments consisted of a layered density model (Fig. 3) with a $P$-wave velocity of 2.9 km/s and an $S$-wave velocity of 1.2 km/s.

Figure 3: Density distribution. The virtual source position $x_F$ is indicated by the white circle; source and receiver positions are represented by triangles.
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Figure 4: (a) Recovered and (b) true vertical velocity component from a P-wave source.

Figure 5: (a) Recovered and (b) true horizontal velocity component from a P-wave source.

Figure 6: (a) Recovered and (b) true vertical velocity component from an S-wave source.

Figure 7: (a) Recovered and (b) true horizontal velocity component from an S-wave source.

$E_0^+$ were modeled using a smooth version of the density. We show the reconstructed and true velocity components from a P-potential source (Figures 4 and 5) and from an S-potential source (Figures 6 and 7). The true and recovered velocity components from an S-wave source are depicted in Figures 6 and 7. Stress components were also recovered but are not shown.

In the P-wave autofocusing we highlight that first and higher order internal multiples are reconstructed, even those that undergo conversion (Figures 4 and 5). Nonconverted events are shown by solid black arrows, while converted events are shown by solid red arrows. We observe that all nonconverted events were recovered faithfully. Moreover, we also observe that converted events were recovered regardless of whether they were converted in the process of transmission (first solid red arrow) or reflection (second solid red arrow), provided they reflected at least once. The only event not reconstructed properly is the P-S conversion of the direct P transmission (dashed black arrow): it does not constitute a reflected event and thus cannot be obtained from correlating reflection data with a direct arrival.

S-wave autofocusing results are similar. Nearly all reflection events (including higher order internal multiples) are reconstructed (solid arrows in Figures 6 and 7). The exception to this are the parts of the first S-P reflection that arrive before the first S arrival (red dashed arrows in Figures 6 and 7); such events are muted by the window $\theta(t + \frac{a^2}{2})$ as they are incorrectly assumed to be nonphysical. This limitation comes from the assumption that all events prior to the first arrival are nonphysical, which is inadequate for S waves. Similarly to what is observed by Vasconcelos et al. (2014) for autofocusing in complex acoustic media, this simple windowing approach may be inadequate. Finally, for reasons explained previously, the converted transmission is also not recovered (black dashed arrow).

CONCLUSION

The first derivation of autofocusing for elastodynamic wavefields using elastic up-down representation theorems and an extension of focusing functions for elastic media is presented. We illustrate the method using a 2D numerical experiment in a layered-density solid medium, and recover Green’s functions from potential sources internal to the medium. Results show accurate reconstructions of nearly all events, with the exception of those that do not satisfy the ansatz; further work is required to overcome this limitation.

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EDITED REFERENCES
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REFERENCES


