frequencies which make the denominator equal to zero. This results in the following expression

$$i\omega\rho_f \frac{J_1(k_r^{(f)}a)}{\partial J_1(k_r^{(f)}a)/\partial a} = i\omega\rho \frac{H_1^{(1)}(k_r^{(c)}a)}{\partial H_1^{(1)}(k_r^{(c)}a)/\partial a}, \tag{4}$$

where $k_r^{(f)}$ is the radial component of the fluid wavenumber, $k_r^{(c)}$ is the radial component of the compressional wavenumber, J_1 is a first order Bessel function, and $H_1^{(1)}$ is a first-order Hankel function of the first type. For all slow formations, as we sweep over all frequencies, equation (4) is never satisfied, indicating that no modes are cutting off. Hence, the flexural mode does not cutoff. It can be shown that equation (4) can be interpreted in terms of radial impedance functions. Specifically, the left-hand side of (4) is the impedance of a standing cylindrical wave in the fluid, while the right-hand side is the impedance of an outward propagating compressional cylindrical wave in the solid. At the frequencies where these two impedances are matched, the modes cutoff. Equation (4) differs from that given in Roever et al. (1974).

In Figure 5 we show the relative strength of the compressional, shear, and flexural modes as a function of frequency. The compressional wave strength at the four offsets considered are the four solid lines, the shear wave strength are the dashed lines. These individual far-field components were computed numerically using a method described in Kurkjian (1985). The flexural mode has the same strength at all offsets and is the dotted line. The decay of the compressional and shear waves with offset can be interpreted as "geometric decay" and has been found to be a function of frequency (Kurkjian and Chang, 1983). We see from Figure 5 that the shear wave dominates at low frequencies, the flexural mode at intermediate frequencies, and the compressional wave at high frequencies. This figure is consistent with Figures 2 and 3 which showed that a 500 Hz source excites a shear wave, and a 4 kHz source excites a flexural mode.

At intermediate frequencies, a useful frequency domain model for the full dipole displacement waveforms is

$$U(z, \omega) \approx V_o(\omega)\ \delta\ R(\omega)\ e^{ik(\omega)z}, \tag{5}$$

where $R(\omega)$ is the excitation of the mode, shown in Figure 5, and $k(\omega) = \omega/v_p(\omega)$. Here $v_p(\omega)$ denotes phase velocity, which was plotted in Figure 4.

In Figure 6 we show synthetic waveforms associated with a 15 kHz source. The amplitude scale is the same as in Figure 3. Here we see only the high frequency Scholte-like part of the flexural mode, plus a dominant compressional wave. The flexural mode component of the waveform is well modeled by equation (5), while the compressional wave requires a more complicated model (Kurkjian, 1985).

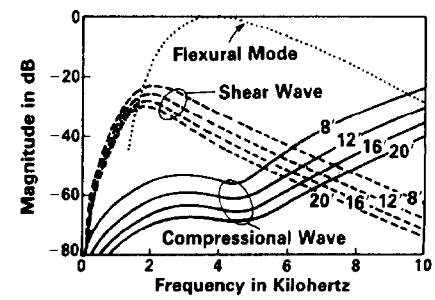

FIG. 5. Relative strength of the compressional, shear, and flexural waves as a function of frequency.

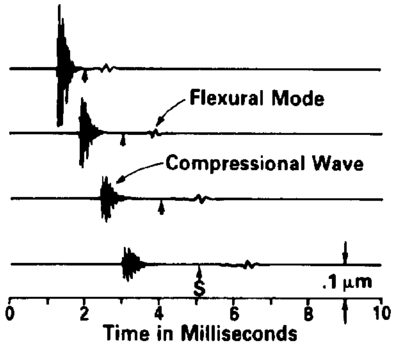

FIG. 6. Synthetic waveforms for a 15 kHz source.

In fast formations, the findings are quite similar to the slow formation with the only exception being the introduction of normal modes at high frequencies. Equation (4) can be used to determine the mode cutoff frequencies.

References

Kurkjian, A. L., 1984, Radiation from a low frequency horizontal acoustic point force in a fluid-filled borehole: Presented at the 54th Annual International SEG Meeting, Atlanta.

———, 1985, Numerical evaluation of individual far-field arrivals excited by an acoustic source in a borehole: Geophysics, **50**, 852–866.

Kurkjian, A. L., and Chang, S. K., 1983, Geometric decay of the head-waves excited by a point force in a fluid-filled borehole: Presented at the 53rd Annual International SEG Meeting, Las Vegas.

Miklowitz, J., 1978, The theory of elastic waves and waveguides: North-Holland Publishing Co., 168.

Roever, W., Rosenbaum, J., and Vining, T., 1974, Acoustic waves from an impulsive source in a fluid-filled borehole: J. Acoust. Soc. Am., **55**, 1144–1157.

Rosenbaum, J. H., 1974, Synthetic microseismograms—logging in porous formations: Geophysics, **39**, 14–32.

Tsang, L., and Rader, D., 1979, Numerical evaluation of transient acoustic waveforms due to a point source in a fluid-filled borehole: Geophysics, **44**, 1706–1720.

The Critical Reflection Theorem S11.2

J. T. Fokkema and A. Ziolkowski, Delft, Univ. of Technology, Netherlands

We prove the following theorem: *The reflection response of a sequence of plane parallel acoustic layers to an incident plane wave that is critically reflected at any interface in the sequence is white.* The proof follows from a consideration of the pressure and normal displacement fields in each layer and the continuity of these fields at the interfaces. At critical incidence at the interface between the nth and $n+1$st layers, the $n+1$st layer acts like a rigid plate. Therefore, all the incident energy is reflected. Since none of the layers is an energy source or sink, the flow of energy in every layer above the $n+1$st layer is the same, independent of frequency. Therefore, at critical incidence, the reflection response is white. The physics of the argument will be identical for the full elastic case, and we expect the theorem to be equally valid for elastic media.

We consider two corollaries of this theorem: (1) At precritical incidence the reflection response is nonwhite. (2) At postcritical incidence at the interface between the nth and $n+1$st layers, the reflection response is white provided the velocity in any of the deeper layers is not less than that in the $n+1$st layer. This theorem may be applied to real seismic reflection data after decomposition into plane waves of varying angles of incidence. The spectra of the postcritically reflected waves are the same as the spectra of the source at the corresponding angles of incidence, provided the earth is elastic.

Proof

We consider a stack of N plane parallel homogeneous acoustic layers, as shown in Figure 1, bounded at the bottom by a homogeneous acoustic halfspace of density ρ_{N+1} and velocity v_{N+1}, and at the top by a homogeneous acoustic half-space of density ρ_o and velocity v_o, where v_o is less than v_{N+1}. A plane pressure wave is incident from the upper half-space traveling parallel to the x-z plane at an angle 0 to the normal to the layers, which is chosen to be the z-axis. We analyze the wave propagation in the space-frequency domain (x, z, ω) and we omit the complex time factor $\exp(-j\omega t)$ in the equations.

The incident field is the plane pressure wave

$$P^{\mathrm{INC}}(x,z,\omega) = A_o^{+}(\omega)\exp\{jk_o(\alpha_o x + \gamma_o z)\} \tag{1}$$

where $A_o^{+}(\omega)$ is the spectrum of the pressure wave function, $k_o = \omega/v_o$ is the wavenumber in the upper half-space, $\alpha_o = \sin^0$, $\gamma_o = \sqrt{1-\alpha_o^2}$. The reflection response of the stratified medium is unknown, but it will be a wave returning at angle 0 to the normal, propagating upwards:

$$P^{R}(x,z,\omega) = A_o^{-}(\omega)\exp\{jk_o(\alpha_o x - \gamma_o z)\}. \tag{2}$$

The total pressure field in the upper half-space is the sum of the incident and reflected fields:

$$P^{o}(x,z,\omega) = \exp(jk_o\alpha_o x)[A^{+}(\omega)\exp(jk_o\gamma_o z) + A_o^{-}(\omega)\exp(-jk_o\gamma_o z)]. \tag{3}$$

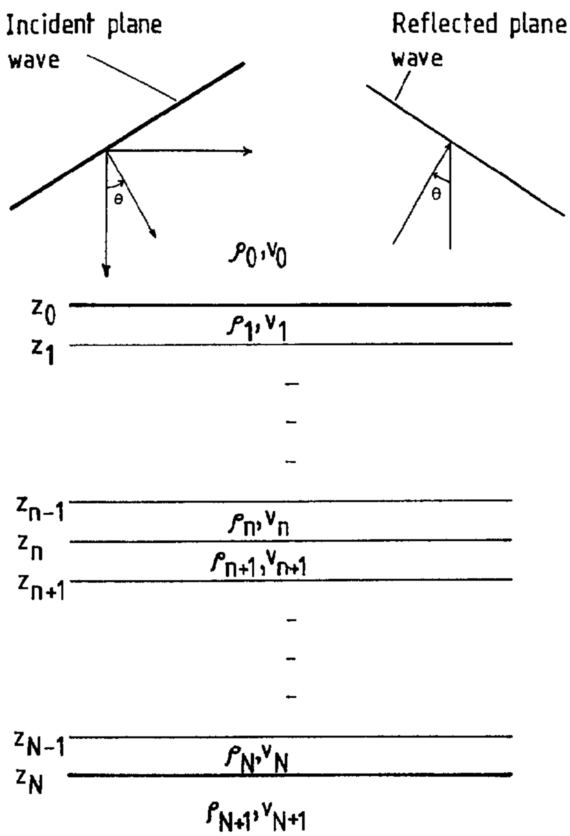

FIG. 1.

The normal component of particle acceleration is related to the pressure gradient by Newton's second law of motion. In the time domain this may be expressed as:

$$\frac{d^2u_z(x,z,t)}{dt^2} = \frac{-1}{\rho}\frac{\partial p(x,z,t)}{\partial z}, \tag{4}$$

where $u_z(x,z,t)$ is the normal component of particle displacement. For our plane wave the total differential with respect to time d/dt is equal to the partial derivative $\partial/\partial t$, and we may transform equation (4) to the frequency domain to yield

$$U_z(x,z,\omega) = \frac{1}{\rho\omega^2}\frac{\partial P(x,z,\omega)}{\partial z}. \tag{5}$$

Thus, in the upper half-space, layer o, we have the normal component of displacement:

$$U_z^0(x,z,\omega) = \frac{jk_o\gamma_o}{\rho_o\omega^2}\exp(jk_o\alpha_o x)\,[A_o^{+}(\omega)\exp(jk_o\gamma_o z) - A_o^{-}(\omega)\exp(-jk_o\gamma_o z)]. \tag{6}$$

Similarly, the pressure and normal displacement fields in the nth and n + 1st layers are

$$p^{n}(x,z,\omega) = \exp(jk_n\alpha_n x)\,[A_n^{+}(\omega)\exp(jk_n\gamma_n z) + A_n^{-}(\omega)\exp(-jk_n\gamma_n z)], \tag{7}$$

$$U_z^{n}(x,z,\omega) = \frac{jk_n\gamma_n}{\rho_n\omega^2}\exp(jk_n\alpha_n x)\,[A_n^{+}(\omega)\exp(jk_n\gamma_n z) - A_n^{-}(\omega)\exp(-jk_n\gamma_n z)], \tag{8}$$

for all values of x and for $z_{n-1} \le z \le z_n$, and

$$p^{n+1}(x,z,\omega) = \exp(jk_{n+1}\gamma_{n+1}x)\,[A_{n+1}^{+}(\omega)\exp(jk_{n+1}\gamma_{n+1}z) + A_{n+1}^{-}(\omega)\exp(-jk_{n+1}\gamma_{n+1}z)] \tag{9}$$

$$U_z^{n+1}(x,z,\omega) = \frac{jk_{n+1}\gamma_{n+1}}{\rho_{n+1}\omega^2}\exp(jk_{n+1}\gamma_{n+1}x)\cdot \cdot[A_{n+1}^{+}(\omega)\exp(-jk_{n+1}\gamma_{n+1}\gamma_{n+1}z) - A_{n+1}^{-}(\omega)\exp(jk_{n+1}\gamma_{n+1}z)], \tag{10}$$

for all values of x and for $z_n \le z \le z_{n+1}$.

From the continuity of pressure at the boundaries for all values of x, we conclude that

$$k_o\alpha_o = k_1\alpha_1 = \cdots = k_n\alpha_n = k_{n+1}\alpha_{n+1} = \cdots k_{N+1}\alpha_{N+1}, \tag{11}$$

which is Snell's law, and can be expressed as

$$\alpha_n = \frac{v_n}{v_o}\sin\theta, \text{ for } n = 0, 1, 2, \ldots, N + 1, \tag{12}$$

from which it is seen that α_n may be greater than 1. Since $\gamma_n = \sqrt{1-\alpha_n^2}$, γ_n will become imaginary if α_n is greater than 1, that is, $\gamma_n = j\sqrt{\alpha_n^2 - 1}$ for $\alpha_n > 1$.

At each boundary we require continuity of pressure and displacement. That is,

$$\lim_{z\uparrow z_n} p^{n}(x,z,\omega) = \lim_{z\downarrow z_n} p^{n+1}(x,z,\omega), \tag{13}$$

$$\lim_{z \uparrow z_n} U_z^n(x,z,\omega) = \lim_{z \downarrow z_n} U_z^{n+1}(x,z,\omega). \quad (14)$$

Let us define a local reflectivity response $R_n(\omega)$ for the nth layer, as follows

$$A_n^-(\omega) = R_n(\omega)A_n^+(\omega)\exp(2jk_n\gamma_n z_n). \quad (15)$$

We now divide equation (14) by equation (13), after substitution from the expressions (7), (8), (9) and (10), and using equation (15) and a similar expression for the n + 1st layer to obtain the following recursion formula for the local reflectivity response:

$$R_n(\omega) = \frac{\Gamma_n + R_{n+1}(\omega)\exp\{2jk_{n+1}\gamma_{n+1}(z_{n+1} - z_n)\}}{1 + \Gamma_n R_{n+1}(\omega)\exp\{2jk_{n+1}\gamma_{n+1}(z_{n+1} - z_n)\}}, \quad (16)$$

in which

$$\Gamma_n = \frac{\gamma_n/\rho_n v_n - \gamma_{n+1}/\rho_{n+1}v_{n+1}}{\gamma_n/\rho_n v_n + \gamma_{n+1}/\gamma_{n+1}v_{n+1}} \quad (17)$$

is the local reflection coefficient. [We immediately recognize that when $\gamma_n = \gamma_{n+1} = 1$, equation (17) is the well-known expression for the reflection coefficient at normal incidence.] Using our definition (15) for the upper half-space, we may rewrite equation (3) as:

$$P_o(x,z,\omega) = A_o^+(\omega)\exp(jk_o\alpha_o x)[\exp(jk_o\gamma_o z) + R_o(\omega)\exp\{jk_o\gamma_o(2z_o - z)\}. \quad (18)$$

At critical incidence at the interface $z = z_n$ we have

$$\alpha_{n+1} = \frac{v_{n+1}}{v_o}\sin\theta = 1,$$

by definition, from which it follows that $\gamma_{n+1} = 0$.

We see immediately by inspection of equation (10) that the particle displacement $U_z(x,z,\omega) = 0$ for all frequencies, for all values of x and for $z_n \le z \le z_{n+1}$. That is, the n + 1st layer acts like a rigid plate at critical incidence, and all the incident energy will be returned to the surface. We see this from the recursion formula (16) and the auxiliary equation (17). When $\gamma_{n+1} = 0$, $\Gamma_n = 1$, and therefore $R_n(\omega) = 1$ for all ω. Now we see that the recursion formula (16) gives

$$R_{n-1}(\omega) = \frac{\Gamma_{n-1} + \exp\{2jk_n\gamma_n(z_n - z_{n-1})\}}{1 + \Gamma_{n-1}\exp\{2jk_n\gamma_n(z_n - z_{n-1})\}}, \quad (19)$$

in which Γ_{n-1} is a real number of modulus less than 1. The numerator and denominator of the right-hand side of equation (19) are complex numbers which have the same modulus, but different phases (see Figure 2). Therefore $R_{n-1}(\omega)$ is a complex number of modulus 1. It follows that $R_{n-2}(\omega)$, $R_{n-3}(\omega)$, . . ., $R_0(\omega)$ are all modulus 1, and from equation (18) we see that the amplitude spectrum of the total field in the upper half-space is the same as the spectrum of the incident pressure wave. Since the total field is simply the sum of the incident field and the reflected field, the reflected field must also have the same amplitude spectrum, and the reflection response of the layered sequence is therefore white.

Corollaries

(1). At precritical incidence Γ_n is real and modulus less than 1 for all layers, and therefore there is a wave transmitted into the lower half-space. There cannot be any upcoming waves in the lower half-space, therefore $R_{N+1}(\omega) = 0$, and we see from the recursion equation (16) that $R_N(\omega) = \Gamma_N$ which has modulus less

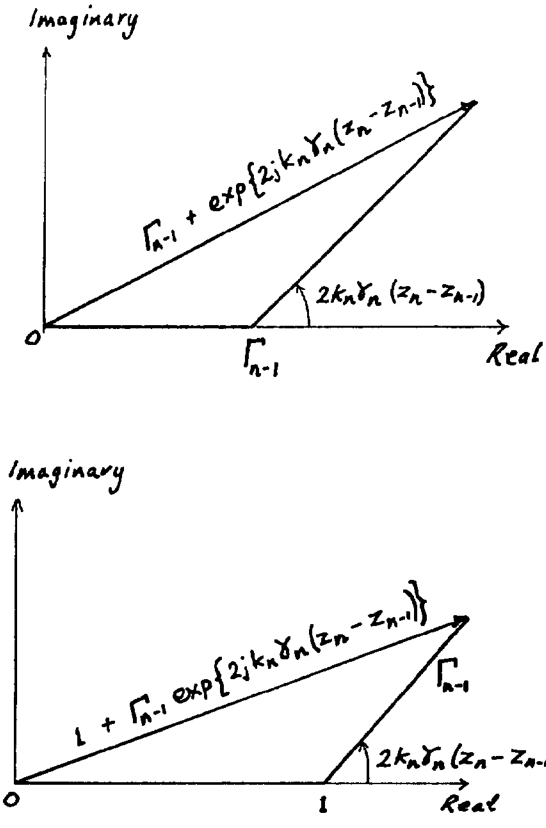

FIG. 2.

than 1 and is real. Using the recursion again, we see that $R_{N-1}(\omega)$ is complex and frequency-dependent, with modulus less than 1. Following the recursion to the top of the stack will all Γ_n real and modulus less than 1, we see that all the $R_n(\omega)$ are complex with modulus less than 1, including $R_0(\omega)$. Therefore the reflection response is not white.

(2). For postcritical incidence at the interface $z = z_n$, we have in the n + 1st layer, $\alpha_{n+1} > 1$ and $\gamma_{n+1} = j\sqrt{\alpha_n^2 - 1}$ and is pure imaginary. Provided none of the velocities in the deeper layers is less than v_{n+1}, it follows that γ_m is pure imaginary for $n + 1 \ge m \ge N + 1$, and Γ_m is real for $n + 1 \ge m \ge N + 1$. Since there are no upcoming waves in the lower half-space $R_n(\omega) = \Gamma_N$, which is real.

At the interface $z = z_n$, where postcritical reflection occurs, γ_n is real and γ_{n+1} is imaginary; therefore Γ_n is complex and modulus 1. From the recursion formula (16) we see that $R_n(\omega)$ is complex and modulus 1 if $R_{n+1}(\omega)$ is real. Using the recursion again, we see that, because Γ_{n+1} is real and γ_{n+2} is imaginary, $R_{n+1}(\omega)$ will be real if $R_{n+2}(\omega)$ is real. And again, $R_{n+2}(\omega)$ will be real if $R_{n+3}(\omega)$ is real, and so on to the bottom of the stack. Since $R_N(\omega)$ *is* real, it follows that $R_m(\omega)$ is real for $n + 1 \ge m \ge N + 1$.

In equation (16), since Γ_n is complex with modulus 1, and $R_{n+1}(\omega)$ is real and γ_{n+1} is imaginary, $R_n(\omega)$ is complex with modulus 1. Using the recursion again, we see that since Γ_{n-1} is real and $R_{n+1}(\omega)$ is modulus 1 and γ_n is real, $R_{n-1}(\omega)$ is complex and modulus 1. Following the recursion upwards, we see that $R_0(\omega)$ is complex and modulus 1. That is, the reflection response is white. Physically, this corollary states that no evanescent waves in the n + 1st layer can result in propagating waves in deeper layers if there is no deeper layer with a velocity less than that of the n + 1st layer.

Conclusions

We can apply this theorem and its corollaries to real data for several purposes and we are sure we have not discovered all the extensions of this theorem. First, the assumption of whiteness for the reflection response cannot be applied before critical incidence. Secondly, at critical incidence and, in many practical cases at postcritical incidence, the plane wave response can be used to determine the spectrum of the incident pressure wave. Thirdly, if the spectrum of this wave is already known, then the spectrum of the postcritical plane wave response will give a check on the linearity of the earth and on the presence of absorbing layers. We illustrate these conclusions with some examples.

The Generalized Primary and the O'Doherty-Anstey Formula S11.3

J. R. Resnick, Western Geophysical; I. Lerche, Univ. of South Carolina; and R. T. Shuey, Chevron Oil Field Research (all formerly with Gulf R&D)

The generalized primary is a signal that leaves the earth's surface, travels to a reflector and returns, dispersing en route in a manner that accounts for all multiples above the reflector. This is a generalization of the Born approximation, for which dispersion is absent. The transmittance of a one-dimensional medium is found by computing the generalized primary at the bottom of the medium. The reflectance of the medium can be expressed as a sum of generalized primary reflections. A straightforward approximation of the generalized primary leads to a formula for the transmitted signal that was first obtained under less general conditions by O'Doherty and Anstey (1971).

Introduction

A single reflection event on a seismic record is often considered to be the result of a process in which a primary wave propagates downward without dispersion, reflects, and returns to the surface. In reality, multiple scattering causes the traveling signal to disperse. This dispersion is usually regarded as a modification to the downgoing signal in the medium, but Richards and Menke (1983, p. 1018) pointed out inconsistencies in this viewpoint. We clarify this issue with an exact formulation of the reflection process in one dimension in terms of a "generalized primary wave." This generalized primary travels to a reflector and returns, but—because it disperses while en route—permits an exact accounting for all multiples. A simple approximation then leads us to a formula for the transmittance of the medium that was first obtained in a more restricted form by O'Doherty and Anstey (1971).

Formulation

We consider a 1-D medium with characteristic impedance $\zeta(x)$, where x is the one-way traveltime from the surface. The reflection coefficient function of the medium is $\rho(x) = \zeta_x(x)/2\zeta(x)$, where the subscript denotes differentiation. With P representing pressure and V signifying particle velocity, define the downgoing signal D and the upcoming signal U as

$$D = \frac{1}{2}(P + \zeta V), \tag{1}$$

$$U = \frac{1}{2}(P - \zeta V). \tag{2}$$

For simplicity we suppose absorbing boundaries exist at the surface $x = 0$ and at the bottom $x = L$. In the frequency domain these conditions are expressed as $D(0,\omega) = 1$, and $U(L,\omega) = 0$. Our goal is to find the transmittance $T(\omega) = D(L,\omega)$ and the reflectance $R(\omega) = U(0,\omega)$.

Riccati equation

Let $H = U/D$. The variable H satisfies the Riccati equation (Schelkunoff, 1951),

$$H_x = \rho H^2 - 2i\omega H - \rho, \tag{3}$$

with the "initial" condition (at the bottom boundary) $H(L,\omega) = 0$. The function D is coupled to H through the relation

$$D(x,\omega) = \exp(i\omega x + \int_0^x \rho(x')\,dx' - \int_0^x \rho(x')H(x',\omega)dx'), \tag{4}$$

and the transmittance follows by putting $x = L$. The reflectance is

$$R(\omega) = H(0,\omega). \tag{5}$$

Invariant imbedding

Consider a medium of length y with $y \le L$, identical to the original for $0 \le x < y$, but having an absorbing boundary at $x = y$. We describe wave propagation in this "truncated" medium using equations (1)-(5), but now we exhibit y as an explicit independent variable, e.g., the Riccati function is written $H(x,y,\omega)$, etc. The reflectance of the original medium can be written

$$R(\omega) = H(0,L,\omega) = \int_0^L H_y(0,y',\omega)dy', \tag{6}$$

where the integral is over the length variable of the truncated medium. This is a variation on the theme of "invariant imbedding" (Bellman and Wing, 1975).

To compute $H_y(0,y,\omega)$, differentiate equation (3) with respect to y, reverse the order of differentiation, and integrate with respect to x. This gives

$$H_y(0,y,\omega) = H_y(y,y,\omega)\exp(2i\omega y - \int_0^y 2\rho(x')H(x',y,\omega)dx'). \tag{7}$$

The boundary condition in the truncated medium is $H(y,y,\omega) = 0$. Hence $H_y(y,y,\omega) = -H_x(y,y,\omega)$, and evaluating equation (3) at $x = y$ yields $H_y(y,y,\omega) = \rho(y)$. Substituting into equation (6), we obtain

$$R(\omega) = \int_0^L \rho(y')\exp(2i\omega y' - \int_0^{y'} 2\rho(x')H(x',y',\omega)dx')dy'. \tag{8}$$

Generalized primary

Define $T_{\text{down}}(y,\omega)$ to be the transmittance from $x = 0$ to $x = y$ of the truncated medium. From equation (4),

$$T_{\text{down}}(y,\omega) = \exp(i\omega y + \int_0^y \rho(x')dx' - \int_0^y \rho(x')H(x',y,\omega)dx'). \tag{9}$$

Also define $T_{\text{up}}(y,\omega)$ as the transmittance from $x = y$ to $x = 0$ of the same medium. This can be shown to be

$$T_{\text{up}}(y,\omega) = \exp(i\omega y - \int_0^y \rho(x')dx' - \int_0^y \rho(x')H(x',y,\omega)dx'). \tag{10}$$